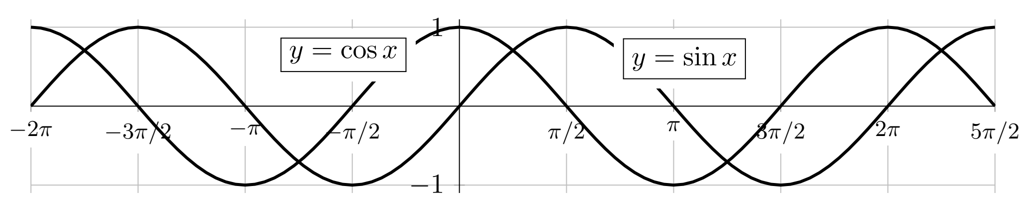
Math 115 — Second Midterm — April 1, 2025

Your Initials Only: ______ Your 8-digit UMID number (not uniqname): ____________________

Instructor Name: ____ Section #: ____________

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 9 pages including this cover. There are 9 problems, which may not all be of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
6. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table, and explain how you used the graph or table to find your answer.
8. Include units in your answer where that is appropriate.
9. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
10. You must use the methods learned in this course to solve all problems.
11. Partial graphs of the sine and cosine functions are included below for convenience.



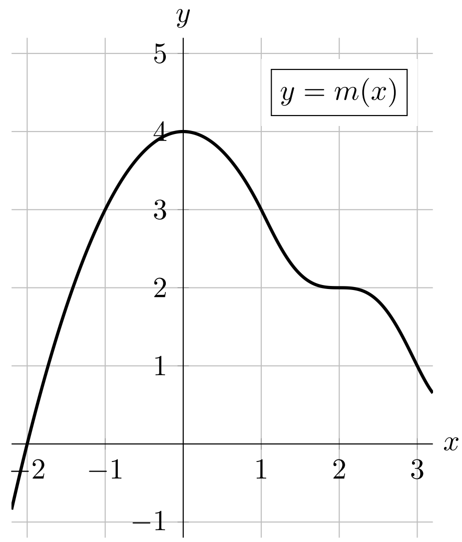
Problem	Points	Score
1	9	
2	10	
3	9	
4	10	
5	14	

Problem	Points	Score
6	4	
7	5	
8	8	
9	11	
Total	80	

1. [9 points]

A portion of the graph of the function $m(x)$, which is defined for all real numbers, is shown to the right. You are also given the following about $m(x)$:

- $m(x)$ is differentiable everywhere, and has a horizontal tangent line at $x = 2$.
- $m(x) = -x^2 + 4$ for all $x \leq 0$.
- The line $y = 5 - 2x$ is tangent to $m(x)$ at $x = 1$.



For parts **a.–c.**, find the **exact** values, or write NEI if there is not enough information to do so, or write DNE if the value does not exist. Your answers should not include the letter m, but you do not need to simplify. *Show work.*

a. [3 points] Let $A(x) = \ln\big(m(x) + x\big)$. Find $A'(-1)$.

Answer: $A'(-1) =$ ________________________

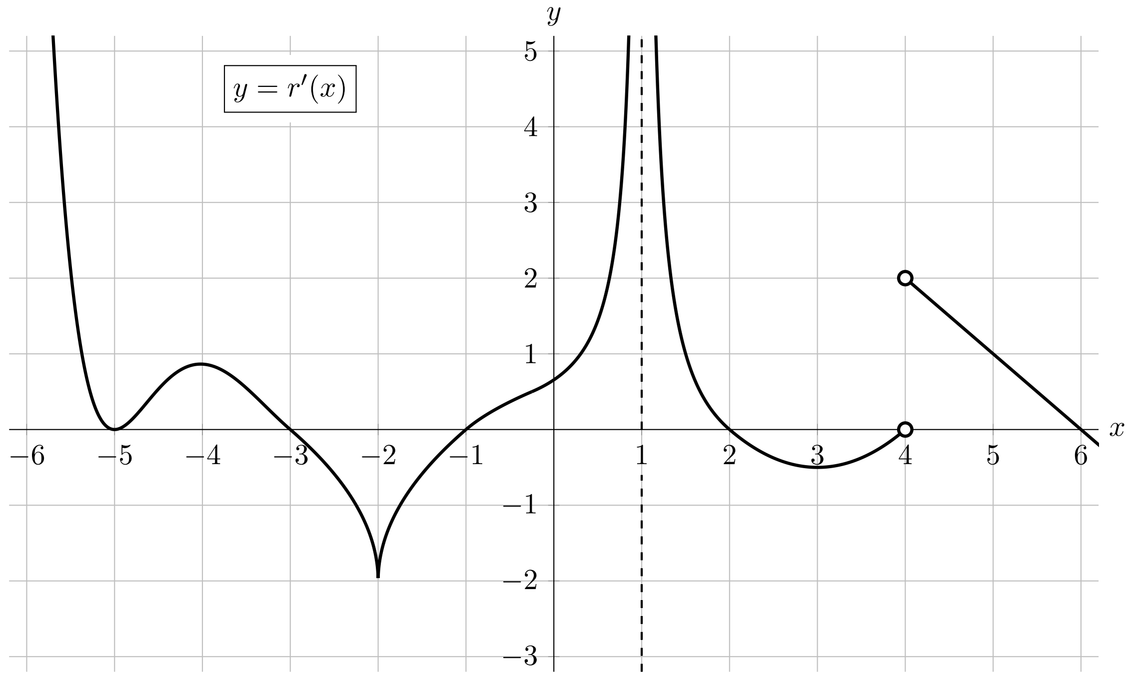
b. [3 points] Let $B(x) = x^3 m(x)$. Find $B'(1)$.

Answer: $B'(1) =$ ________________________

c. [3 points] Let $C(x) = \dfrac{m(x)}{x^2}$. Find $C'(2)$.

Answer: $C'(2) =$ ________________________

2. [10 points] Suppose $r(x)$ is a continuous function, defined for all real numbers. A portion of the graph of $r'(x)$, the **derivative** of $r(x)$, is given below. Note that $r'(x)$ has a vertical asymptote at $x = 1$ and a sharp corner at $x = -2$, and is undefined only at $x = 1$ and $x = 4$.



a. [2 points] Circle all points below that are critical points of $r(x)$.

$x = -5$ $x = -3$ $x = -2$ $x = 1$ $x = 3$ NONE OF THESE

b. [2 points] Circle all points below that are local maxima of $r(x)$.

$x = -5$ $x = -3$ $x = -1$ $x = 1$ $x = 4$ NONE OF THESE

c. [2 points] Circle all points below that are local minima of $r(x)$.

$x = -5$ $x = -3$ $x = -1$ $x = 1$ $x = 4$ NONE OF THESE

d. [2 points] Circle all points below that are inflection points of $r(x)$.

$x = -5$ $x = -4$ $x = -2$ $x = 2$ $x = 4$ NONE OF THESE

e. [2 points] Circle all intervals below on which $r'(x)$ satisfies the hypotheses of the Mean Value Theorem.

$[-5, -3]$ $[-3, -1]$ $[-2, 0]$ $[0, 2]$ $[2, 4]$ NONE OF THESE

3. [9 points] A factory makes cylindrical cans of volume 400 cubic centimeters. Suppose the metal for the side of the can costs 1 cent per cm^2, and the metal for the top and the bottom costs 2 cents per cm^2. Find the **radius** of the can shape that minimizes the cost of producing such a can.

Show all your work, include units, and fully justify using calculus that you have in fact found the radius that minimizes cost.

Answer: radius = ______________________

4. [10 points] Let $f(x)$ be the differentiable function defined by

$$f(x) = x^3 + \cos(x^3), \qquad \text{so} \qquad f'(x) = 3x^2\big(1 - \sin(x^3)\big).$$

For each part below, you must use calculus to find and justify your answers. Clearly state your conclusions and show enough evidence to support them. You may use the graphs of sine and cosine given on the front page, if necessary. Recall that $\pi \approx 3.14$.

a. [3 points] The function $f(x)$ has exactly <u>three</u> critical points **in the interval** $(-1, 2)$. Find them. Give exact answers, and *show your work.*

Answer: $f(x)$ has critical points at $x =$ ____________________

b. [4 points] Find the x-coordinates of all *local* minima and maxima of $f(x)$ **on the interval** $(-1, 2)$. If there are none of a particular type, write NONE. *Justify your answers.*

Answer: Local min(s) at $x =$ ____________ and Local max(es) at $x =$ ____________

c. [3 points] Find the x-coordinates of all *global* minima and maxima of $f(x)$ **on the interval** $[-1, 1]$. If there are none of a particular type, write NONE.

Answer: Global min(s) at $x =$ ____________ and Global max(es) at $x =$ ____________

5. [14 points] Throughout this problem, let $g(x)$ and $h(x)$ be the functions defined by

$$g(x) = 2x^2 + e^{(x^3)} \quad \text{and} \quad h(x) = e^C + \ln(x^k),$$

where C and k are positive constants.

a. [4 points] Compute the derivatives of $g(x)$ and $h(x)$, remembering that C and k are **constants**. *Show your work.*

Answer: $g'(x) =$ ____________________ **Answer:** $h'(x) =$ ____________________

b. [2 points] Find a formula for the linear approximation $L(x)$ of the function $g(x)$ at the point $x = 1$. Your answer should not include the letter g.

Answer: $L(x) =$ ______________________________

c. [4 points] There exist values of the constants C and k for which the piecewise function

$$f(x) = \begin{cases} g(x) & x \leq 1 \\ h(x) & x > 1 \end{cases}$$

is continuous and differentiable. Find such values of C and k, and *show all your work.*

Answer: $C =$ ____________________ **Answer:** $k =$ ____________________

Problem 5 continues on the next page.

Problem 5 continues from the previous page. Recall that

$$f(x) = \begin{cases} g(x) & x \leq 1 \\ h(x) & x > 1 \end{cases}$$

and $L(x)$ is the linear approximation of $g(x)$ at $x = 1$. For part **d**. below, let C and k be the constants that you found in part **c**., so $f(x)$ is continuous and differentiable.

d. [4 points] You are given that $g''(x) > 0$ on the domain of $g(x)$, while $h''(x) < 0$ on the domain of $h(x)$. Using this, answer the questions below, and *justify each answer with a brief explanation.*

i. Does the function $L(x)$ from part **b.** give an overestimate or underestimate for $g(x)$ near $x = 1$? Circle your answer, and briefly justify it.

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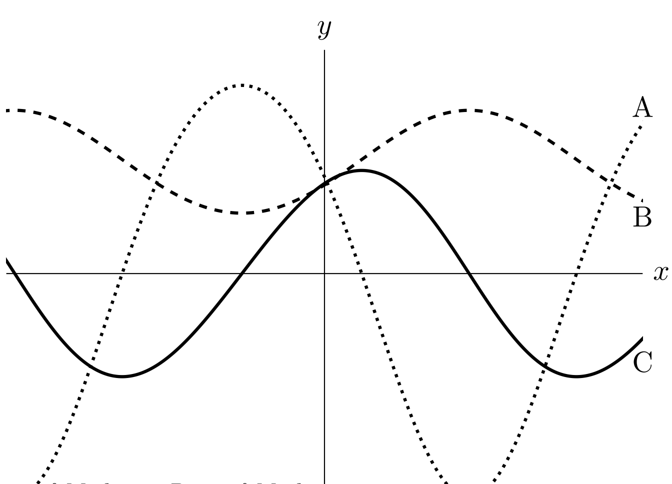
ii. List the x-values of all inflection points of $f(x)$, or write NONE if $f(x)$ has no inflection points. Briefly justify your answer.

Answer: $x =$ ____________________

6. [4 points] Shown below are portions of the graphs of the functions $y = f(x)$, $y = f'(x)$, and $y = f''(x)$. Determine which graph is which, and then, on the answer lines below, indicate after each function the letter A, B, or C that corresponds to its graph. *No work or justification is needed.*



Answer: $f(x)$: __________

$f'(x)$: __________

$f''(x)$: __________

7. [5 points] The equation $x^3 + y^3 - xy^2 = 5$ defines y implicitly as a function of x. Find a formula for $\frac{dy}{dx}$ in terms of x and y. Show every step of your work.

Answer: $\frac{dy}{dx} =$ ______________________

8. [8 points] Let $\mathcal{C}$ be the curve defined by the equation $x^2 + y^3 = 8y$. Note that

$$\frac{dy}{dx} = \frac{2x}{8 - 3y^2}.$$

a. [4 points] Find the coordinates of all points (x, y) on the curve $\mathcal{C}$ where the tangent line to $\mathcal{C}$ is horizontal. Write your answer as a list of points in the form (x, y), or write NONE if there are no such points. *Show all your work.*

Answer: ______________________

b. [4 points] The curve $\mathcal{C}$ intersects the line $y = 1$ at exactly one point with a positive x value. Find an equation of the line tangent to the curve $\mathcal{C}$ at this point. *Show all your work.*

Answer: $y =$ ______________________

9. [11 points] A spherical balloon begins to inflate with air at time $t = 0$, after which time its radius r, volume V, and surface area A increase. Recall that the volume V and surface area A of a sphere of radius r are given by $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$.

a. [5 points] At what rate is air being blown into the balloon when the balloon's radius is 10 cm and its radius is growing at a rate of 2 cm per second? *Include units.*

Answer: ______________________________

Suppose the volume V and surface area A of the balloon t seconds after it begins to inflate are given by $V = f(t)$ and $A = g(t)$. These functions are invertible, and the function $h(V)$ defined by $h(V) = g\big(f^{-1}(V)\big)$ gives the balloon's surface area as a function of its volume.

b. [3 points] Using the given table of values, find $h'(8)$. Your answer must be a *number*, but need not be simplified.

t	1	8
r	1.24	2.48
$f(t)$	8	64
$g(t)$	19.34	77.38
dr/dt	0.41	0.10
$f'(t)$	8	8
$g'(t)$	12.90	6.45

Answer: $h'(8) =$ ______________________________

c. [3 points] Circle the **one** statement below that is best supported by the equation

$$(h^{-1})'(50) = \frac{1}{4}.$$

i. When the balloon's surface area is 50 cm^2, its volume is 0.25 cm^3.

ii. When the balloon's surface area is 50 cm^2, its surface area is increasing at about $\frac{1}{4}$ the rate at which its volume is increasing.

iii. When the balloon's surface area is 52 cm^2, the balloon is about one cubic centimeter larger in volume than it was when its surface area was 48 cm^2.

iv. The balloon's surface area increases by about 0.25 cm^2 during the time when its volume increases from 50 to 51 cm^3.

v. The balloon's volume increases by about 4 cm^3 during the time when its surface area increases from 50 to 51 cm^2.