Math 115 — First Midterm — February 11, 2025

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 7 pages including this cover. There are 8 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
   No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	11	
2	9	
3	4	
4	5	

Problem	Points	Score
5	7	
6	6	
7	10	
8	8	
Total	60	

1. [11 points] The temperature in degrees Celsius ($^\circ$C) of a certain cup of hot tea x minutes after it has been poured is given by $T(x) = 20 + 60\left(\frac{1}{2}\right)^{x/30}$.

a. [2 points]

i. What was the initial temperature of the tea in degrees Celsius?

Solution: $T(0) = 20 + 60\left(\frac{1}{2}\right)^{0}/30 = 20 + 60(1) = 80$.

Answer: 80 $^\circ$C

ii. What is $\lim_{x\to\infty} T(x)$?

Solution: Since $\lim_{x\to\infty}(1/2)^{x/30} = 0$, we have $\lim_{x\to\infty} T(x) = 20 + 0 = 20$.

Answer: 20

b. [3 points] The hot tea sits on a table cooling for an entire hour before you remember to drink it. Find the average rate at which the tea cools during this hour. *Your answer should be a positive number. Include units.*

Solution: The average rate of *change* of the temperature of the tea over the first 60 minutes after it was poured is

$$\frac{T(60)-T(0)}{60-0} = \frac{20+\frac{60}{4}-80}{60} = \frac{15-60}{60} = -\frac{45}{60} = -\frac{3}{4} \quad \text{degrees Celsius per minute.}$$

Since we want the average rate of *decrease*, we need to take the absolute value of this number.

Answer: $\frac{3}{4}^\circ$ **C per minute, or** 45° **C per hour**

c. [3 points] If t is the temperature of a liquid in degrees Celsius, then its temperature in degrees Fahrenheit is $f(t) = \frac{9}{5}t + 32$.

i. [2 points] Find constants m and b such that $f^{-1}(x) = mx + b$.

Solution: We set $x = \frac{9}{5}t + 32$ and solve for t as a function of x:

$$x - 32 = \frac{9}{5}t, \quad \text{so} \quad f^{-1}(x) = t = \frac{5}{9}(x-32) = \frac{5}{9}x - \frac{5\cdot 32}{9} = \frac{5}{9}x - \frac{160}{9}$$

Answers: $m =$ 5/9 $b =$ $-160/9$

ii. [1 point] Write an expression for the temperature of the tea in degrees Fahrenheit x minutes after it has been poured. *Your answer may involve one or both of the letters T or f, but it does not have to; either way, you do not need to simplify.*

Answer: $f\big(T(x)\big)$, **or** $\frac{9}{5}\left(20 + 60\left(\frac{1}{2}\right)^{x/30}\right) + 32$

d. [3 points] Assuming the tea was poured at 12 noon, circle the **one best** practical interpretation of the fact that $(T^{-1})'(50) \approx -1.5$.

i. At 12:50 pm, the tea is cooling at a rate of about 1.5° C per minute.

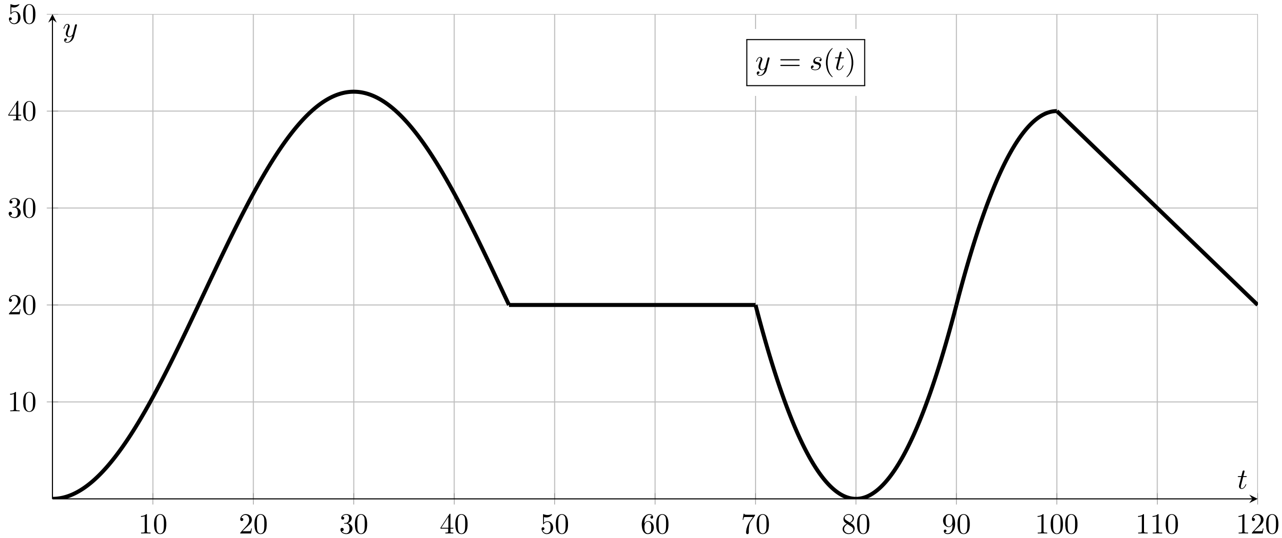
$\boxed{ii.}$ It takes about three minutes for the tea to cool down from 51° C to 49° C.

iii. 50 minutes after the tea was poured, it takes the tea about 90 seconds to cool down 1° C.

iv. It takes about a minute for the tea to cool down from 51.5° C to 50° C.

v. The tea had a temperature of 60° C fifteen minutes before its temperature was 50° C.

2. [9 points] You are flying your new drone over Gallup Park. On this first test flight, you just practice flying straight up and down, directly above the launch point. Suppose $s(t)$ gives the drone's vertical height above the ground, in meters, t seconds after being launched. A graph of $s(t)$ is given below. Note that $s(t)$ is linear on the intervals $[45, 70]$ and $[100, 120]$.



a. [2 points] During which of the following time intervals is the rate of change of the drone's height constant? Circle all correct choices.

$(0, 20)$ $(20, 40)$ $\boxed{(50, 70)}$ $\boxed{(110, 120)}$ NONE OF THESE

b. [1 point] At which of the following times did the drone have the greatest instantaneous veclocity? Circle the one correct answer.

$t = 10$ $t = 30$ $t = 75$ $t = 80$ $\boxed{t = 90}$

c. [1 point] At which time t in the interval $[0, 70]$ was the drone's instantaneous velocity closest to its average velocity over the time interval $[0, 70]$? *Circle the one best answer.*

$t = 10$ $t = 20$ $\boxed{t = 29}$ $t = 31$ $t = 40$

d. [2 points] Find the drone's *instantaneous velocity* 110 seconds after being launched. *Include units.*

Solution: The drone's instantaneous velocity at $t = 110$ is the slope of the graph of $s(t)$ at $t = 110$, which is -1. And since y is in meters and t is in seconds, the units are meters per second.

Answer: **-1 meter per second**

e. [3 points] Find the drone's *average speed* over the time interval $[50, 80]$. *Include units.*

Solution: The drone's average speed over the time interval $[50, 80]$ is the absolute value of

$$\frac{s(80) - s(50)}{80 - 50} = \frac{0 - 20}{30} = -\frac{2}{3} \text{ m/s.}$$

Answer: **2/3 meters per second**

3. [4 points] An insect population in a certain large park varies sinusoidally from a low of 10 million on January 1st to a high of 70 million on July 1st. Let $P(t)$ be the population in the park of this insect, in millions, t months after January 1st. Find a formula for $P(t)$.

Solution: We are looking for a sinusoidal function, of the form $y = A\sin\big(B(t-h)\big) + C$ or $y = A\cos\big(B(t-h)\big) + C$. Since $P(0)$ is a minimum, we will use cosine instead of sine, with A negative, and we do not need any phase shift so $h = 0$. Since $P(t)$ varies between 10 and 70, its amplitude will be $\frac{70-10}{2} = 30$, and the midline will be $\frac{10+70}{2} = 40$. So $A = -30$ and $C = 40$. Finally, since the period is 12 months, we have $B = \frac{2\pi}{12}$. Therefore,

$$P(t) \;=\; -30\cos\left(\frac{2\pi t}{12}\right) + 40.$$

Answer: $P(t) =$ $-30\cos\left(\frac{2\pi t}{12}\right) + 40$

4. [5 points] Let

$$g(x) = \begin{cases} \dfrac{\arctan x}{x} & x \neq 0, \\ 1 & x = 0. \end{cases}$$

You are given that $g'(0)$ exists. Use the limit definition of the derivative to write an explicit expression for $g'(0)$. *Your answer should not involve the letter g. Do not attempt to evaluate or simplify the limit.* Write your final answer in the answer box provided below.

Solution:

$$g'(0) \;=\; \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} \;=\; \lim_{h\to 0} \frac{\frac{\arctan(0+h)}{0+h} - 1}{h} \;=\; \lim_{h\to 0} \frac{\frac{\arctan(h)}{h} - 1}{h}.$$

Answer: $g'(0) =$ $\boxed{\lim_{h\to 0} \frac{\frac{\arctan(h)}{h} - 1}{h}}$

5. [7 points] Consider the rational functions

$$p(x) = \frac{x(x+3)^3}{7(x-4)^2} \qquad \text{and} \qquad q(x) = \frac{5(x-2)(x-4)}{(x+3)(x+1)(x-2)},$$

and let $R(x) = p(x) \cdot q(x)$ be their product. In (a)–(d) below, circle *all* correct answers or else NONE OF THESE if there are no correct answers.

a. [1 point] Which of the following points belong to the domain of $R(x)$?

$x=-3$ $\quad$ $x=-1$ $\quad$ $\boxed{x=0}$ $\quad$ $x=2$ $\quad$ $x=4$ $\quad$ NONE OF THESE

b. [1 point] At which of the following points does $R(x)$ have a vertical asymptote?

$x=-3$ $\quad$ $\boxed{x=-1}$ $\quad$ $x=0$ $\quad$ $x=2$ $\quad$ $\boxed{x=4}$ $\quad$ NONE OF THESE

c. [1 point] At which of the following points does $R(x)$ have a hole?

$\boxed{x=-3}$ $\quad$ $x=-1$ $\quad$ $x=0$ $\quad$ $\boxed{x=2}$ $\quad$ $x=4$ $\quad$ NONE OF THESE

d. [1 point] Which of the following are horizontal asymptotes of $R(x)$?

$y=5$ $\quad$ $y=7$ $\quad$ $y=\frac{5}{7}$ $\quad$ $y=\frac{7}{5}$ $\quad$ $y=0$ $\quad$ $\boxed{\text{NONE OF THESE}}$

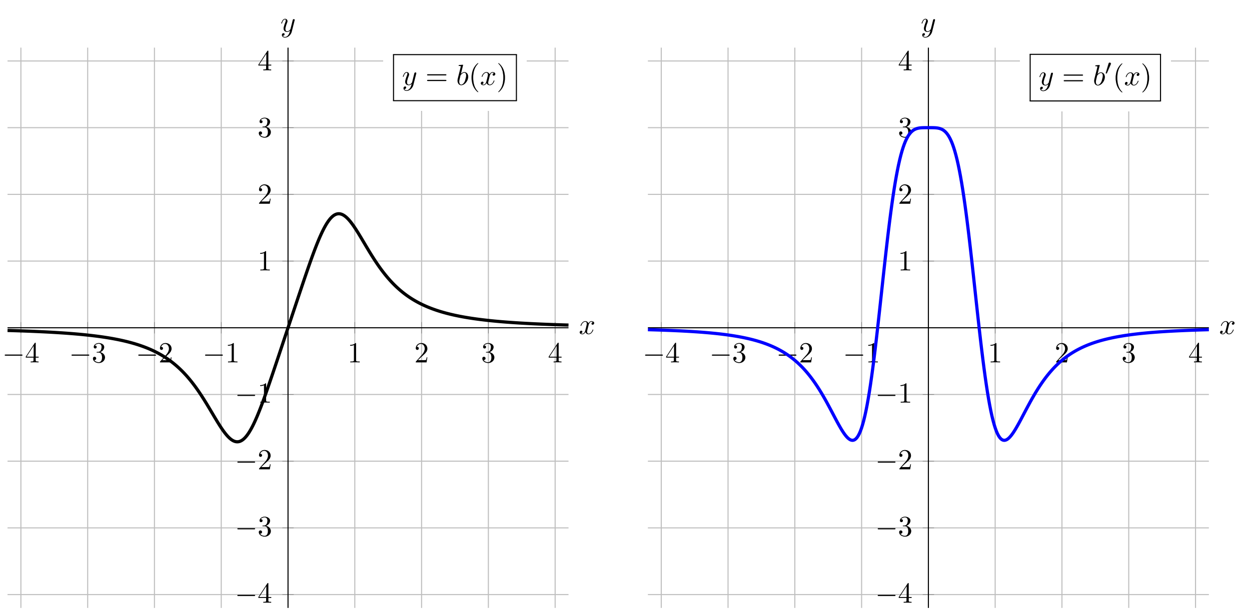
e. [3 points] Compute the following limits, writing DNE if a given limit does not exist.

(i) $\lim_{x \to -3} p(x) =$ 0

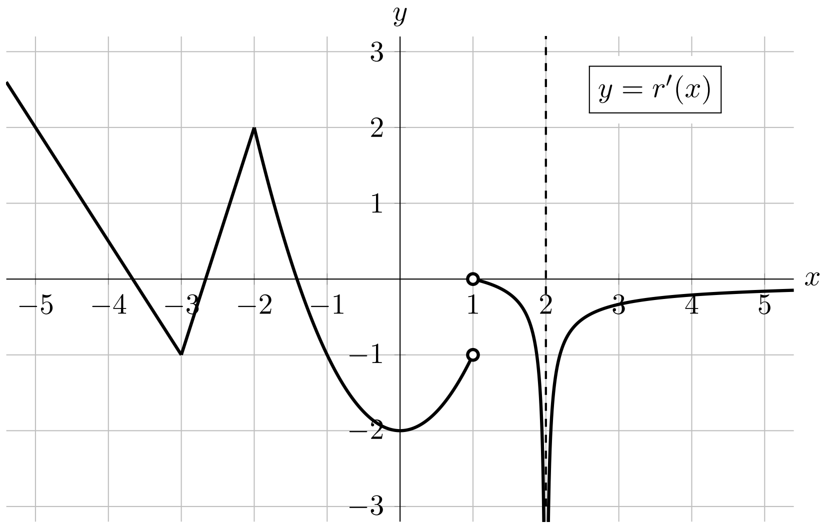
(ii) $\lim_{x \to -3} q(x) =$ **DNE**

(iii) $\lim_{x \to -3} R(x) =$ 0

6. [6 points] A portion of the graph of the function $b(x)$ is shown below on the left. Carefully sketch the graph of the derivative $b'(x)$ of $b(x)$ for $-4 < x < 4$ on the given axes on the right.



7. [10 points] Given below is a portion of the graph of $r'(x)$, the derivative of the continuous function $r(x)$, along with a table of some values of $r(x)$. Note that $r'(x)$ has a vertical asymptote at $x = 2$. Use the graph and the table to answer the questions below. *You do not need to show work.*



x	-3	-2	1	2
$r(x)$	6.5	7	4	??

a. [1 point] Circle all of the x values below at which the function $r'(x)$ is not continuous.

$x = -2$ $\quad$ $x = 0$ $\quad$ $\boxed{x = 1}$ $\quad$ $\boxed{x = 2}$ $\quad$ NONE OF THESE

b. [6 points] Find the **exact** numerical value of each expression below, if possible. For any values that do not exist, including if they are limits that diverge to $\pm\infty$, write DNE.

i. $\displaystyle\lim_{x\to 0} r'(x) =$ -2

iv. $\displaystyle\lim_{x\to -1} r'(2x+3) =$ **DNE**

ii. $\displaystyle\lim_{x\to 1^-} r'(x) =$ -1

v. $\displaystyle\lim_{h\to 0} \frac{r'(-4+h) - r'(-4)}{h} =$ $-3/2$

iii. $\displaystyle\lim_{x\to 2^+} \frac{1}{r'(x)} =$ 0

vi. $\displaystyle\lim_{t\to 0} \frac{r(-2+t) - 7}{t} =$ 2

c. [1 point] Given that $r(2)$ is one of the five values below, determine which one it is by circling the one correct answer.

$\boxed{\frac{10}{3}}$ $\quad$ 4 $\quad$ 5 $\quad$ $\frac{16}{3}$ $\quad$ $4 + 2^{1/3}$

d. [2 points] Find an equation of the line tangent to the graph of $r(x)$ at $x = -3$.

Solution:

$$y = r(-3) + r'(-3)(x+3) = 6.5 + (-1)(x+3) = 3.5 - x.$$

Answer: $y =$ $6.5 - (x+3)$, **or** $y = 3.5 - x$

8. [8 points] Sound intensity, or loudness, as measured in decibels, decreases as you move further away from a sound source. If you move from a distance of a meters away from a certain sound source to a new distance of b meters away from it, then the sound intensity changes by

$$20 \log \left(\frac{a}{b}\right) \text{ decibels.}$$

a. [3 points] Suppose a certain sound source has an intensity of 97 decibels when you stand 9 meters away from it. How far away from the sound source do you need to stand in order to reduce its sound intensity to 90 decibels? *Show all your work, and give your answer in exact form.*

Solution: We are initially standing $a = 9$ meters from the sound souce, and need to determine a new distance b meters away from the sound source so that the loudness changes by $90 - 97 = -7$ decibels. So we need to solve the equation

$$20 \log \left(\frac{9}{b}\right) = -7$$

for b. Solving this gives us:

$$\log \left(\frac{9}{b}\right) = -\frac{7}{20}, \qquad \frac{9}{b} = 10^{-7/20}, \qquad b = \frac{9}{10^{-7/20}} = 9 \cdot 10^{7/20}.$$

Answer: $9/(10^{-7/20}) \ = \ 9 \cdot 10^{7/20}$ meters

b. [3 points] Write a sentence that gives a practical interpretation, in the context of sound intensity, of the following statement:

$$\text{for any positive number } d, \quad 20 \log \left(\frac{d}{10d}\right) = -20.$$

Your sentence should *not* include the variable d.

Solution: Moving 10 times as far away from a sound source decreases its loudness by 20 decibels.

c. [2 points] Suppose the sound intensity S in decibels at a game in the Big House is given as a function of crowd attendance by $S = f(p)$, where p is the number of fans in attendance.

Complete the sentence below to give a practical interpretation of the equation

$$f'(90{,}000) = 10^{-4}.$$

When there are __90,000__ *fans in the Big House, you have to add about* __10,000__ *more fans in order to increase crowd noise by 1 decibel.*