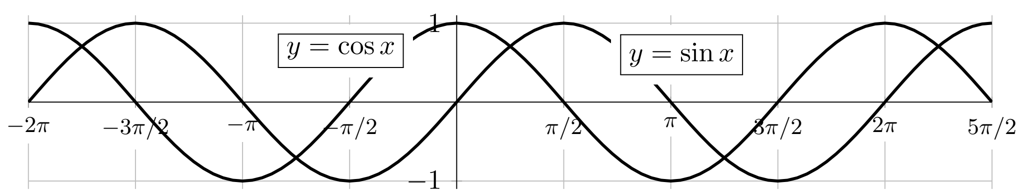
Math 115 — Second Midterm — April 1, 2025

EXAM SOLUTIONS

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 9 pages including this cover. There are 9 problems, which may not all be of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
6. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table, and explain how you used the graph or table to find your answer.
8. Include units in your answer where that is appropriate.
9. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
10. You must use the methods learned in this course to solve all problems.
11. Partial graphs of the sine and cosine functions are included below for convenience.



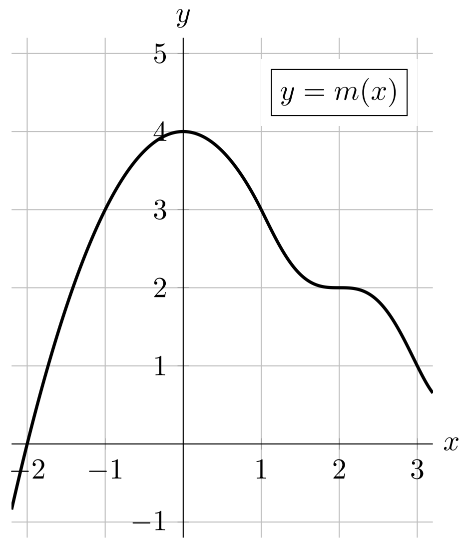
Problem	Points	Score
1	9	
2	10	
3	9	
4	10	
5	14	

Problem	Points	Score
6	4	
7	5	
8	8	
9	11	
Total	80	

1. [9 points]

A portion of the graph of the function $m(x)$, which is defined for all real numbers, is shown to the right. You are also given the following about $m(x)$:

- $m(x)$ is differentiable everywhere, and has a horizontal tangent line at $x = 2$.
- $m(x) = -x^2 + 4$ for all $x \leq 0$.
- The line $y = 5 - 2x$ is tangent to $m(x)$ at $x = 1$.



For parts **a.–c.**, find the **exact** values, or write NEI if there is not enough information to do so, or write DNE if the value does not exist. Your answers should not include the letter m, but you do not need to simplify. *Show work.*

a. [3 points] Let $A(x) = \ln\big(m(x) + x\big)$. Find $A'(-1)$.

Solution: From the graph we see $m(-1) = 3$, and since $m(x) = -x^2 + 4$ for $x \leq 0$, we have $m'(-1) = -2(-1) = 2$. Therefore, by the Chain Rule and the rule for differentiating the natural log function, we have

$$A'(x) = \frac{m'(x) + 1}{m(x) + x}, \quad \text{so} \quad A'(-1) = \frac{m'(-1) + 1}{m(-1) - 1} = \frac{2 + 1}{3 - 1} = \frac{3}{2}.$$

Answer: $A'(-1) =$ $3/2$

b. [3 points] Let $B(x) = x^3 m(x)$. Find $B'(1)$.

Solution: From the graph we see $m(1) = 3$, and since the line $y = 5 - 2x$ is tangent to $m(x)$ at $x = 1$, we have $m'(1) = -2$. Therefore, by the Product Rule we have

$$B'(x) = 3x^2 m(x) + x^3 m'(x), \quad \text{so} \quad B'(1) = 3m(1) + m'(1) = 9 + (-2) = 7.$$

Answer: $B'(1) =$ 7

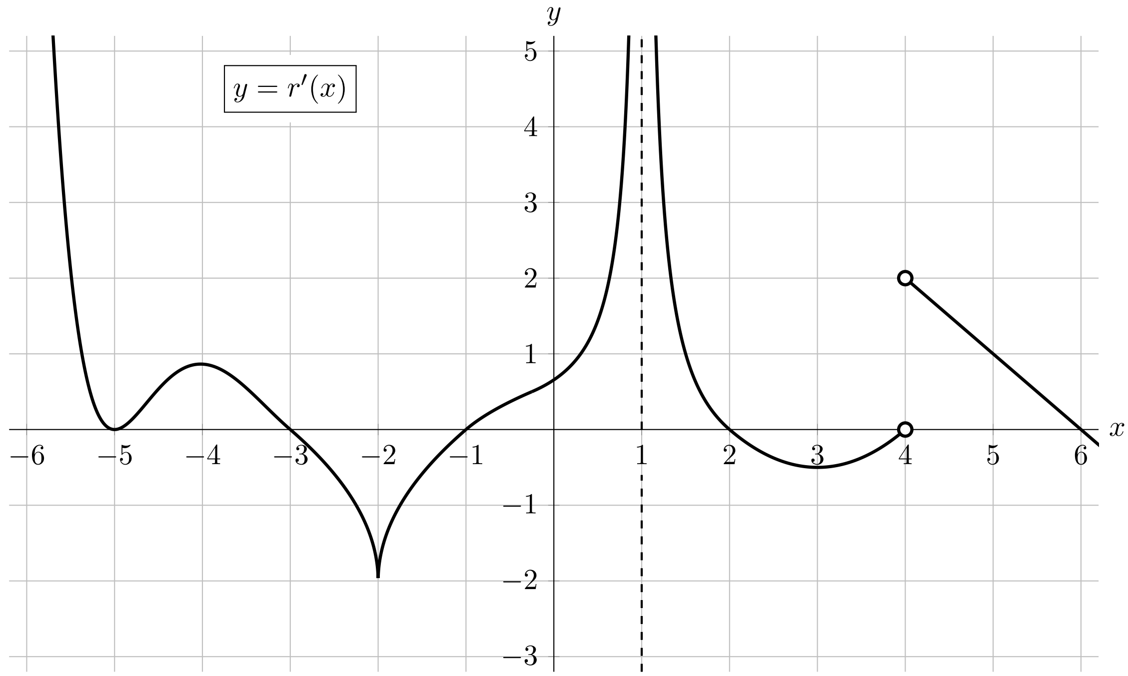
c. [3 points] Let $C(x) = \dfrac{m(x)}{x^2}$. Find $C'(2)$.

Solution: From the graph we see $m(2) = 2$ and $m'(2) = 0$, so by the Quotient Rule we have

$$C'(x) = \frac{m'(x)x^2 - 2x m(x)}{x^4}, \quad \text{so} \quad C'(2) = \frac{4m'(2) - 4m(2)}{2^4} = \frac{0 - 4(2)}{16} = -\frac{1}{2}.$$

Answer: $C'(2) =$ $-1/2$

2. [10 points] Suppose $r(x)$ is a continuous function, defined for all real numbers. A portion of the graph of $r'(x)$, the **derivative** of $r(x)$, is given below. Note that $r'(x)$ has a vertical asymptote at $x = 1$ and a sharp corner at $x = -2$, and is undefined only at $x = 1$ and $x = 4$.



a. [2 points] Circle all points below that are critical points of $r(x)$.

$\boxed{x = -5}$ $\boxed{x = -3}$ $x = -2$ $\boxed{x = 1}$ $x = 3$ NONE OF THESE

b. [2 points] Circle all points below that are local maxima of $r(x)$.

$x = -5$ $\boxed{x = -3}$ $x = -1$ $x = 1$ $x = 4$ NONE OF THESE

c. [2 points] Circle all points below that are local minima of $r(x)$.

$x = -5$ $x = -3$ $\boxed{x = -1}$ $x = 1$ $\boxed{x = 4}$ NONE OF THESE

d. [2 points] Circle all points below that are inflection points of $r(x)$.

$\boxed{x = -5}$ $\boxed{x = -4}$ $\boxed{x = -2}$ $x = 2$ $\boxed{x = 4}$ NONE OF THESE

e. [2 points] Circle all intervals below on which $r'(x)$ satisfies the hypotheses of the Mean Value Theorem.

$\boxed{[-5, -3]}$ $[-3, -1]$ $\boxed{[-2, 0]}$ $[0, 2]$ $[2, 4]$ NONE OF THESE

3. [9 points] A factory makes cylindrical cans of volume 400 cubic centimeters. Suppose the metal for the side of the can costs 1 cent per cm^2, and the metal for the top and the bottom costs 2 cents per cm^2. Find the **radius** of the can shape that minimizes the cost of producing such a can.

Show all your work, include units, and fully justify using calculus that you have in fact found the radius that minimizes cost.

Solution: Let r be the radius of the can, and h its height, both in centimeters. The volume of the can is then $\pi r^2 h$, so we obtain the constraint equation

$$400 = \pi r^2 h, \qquad \text{or} \qquad h = \frac{400}{\pi r^2} \tag{1}$$

after solving for h. Let C be the cost of producing a single can, in cents, so

$$C = 2\pi r h + 4\pi r^2 \tag{2}$$

since the cost of producing the side of a can is 1 cent times the surface area $2\pi r h$, while the cost of producing each of the top and bottom separately is 2 cents times the surface area πr^2.
We want to minimize C, so we use Equation (1) to rewrite our expression for C in terms of one variable. Substituting $\frac{400}{\pi r^2}$ for h in Equation (2) gives

$$C \;=\; C(r) \;=\; 2\pi r\left(\frac{400}{\pi r^2}\right) + 4\pi r^2 \;=\; \frac{800}{r} + 4\pi r^2. \tag{3}$$

The radius of the can must be positive, so we want to minimize the function $C = \frac{800}{\pi} + 4\pi r^2$ on the domain $(0, \infty)$. Differentiating with respect to r gives us

$$\frac{dC}{dr} \;=\; \frac{-800}{r^2} + 8\pi r. \tag{4}$$

Setting this derivative equal to 0 and solving for r, we get

$$\frac{-800}{r^2} + 8\pi r = 0, \qquad \text{so} \qquad 8\pi r^3 = 800, \qquad \text{thus} \qquad r = \left(\frac{100}{\pi}\right)^{1/3}.$$

So $r = \left(\frac{100}{\pi}\right)^{1/3}$ is the only critical point of $C(r)$. To verify that it is a global minimum of $C(h)$ on $(0, \infty)$, we note that

$$\lim_{x\to 0^+} C(r) \;=\; \lim_{x\to 0^+}\left(\frac{800}{r} + 4\pi r^2\right) \;=\; \infty,$$

and likewise

$$\lim_{x\to \infty} C(r) \;=\; \lim_{r\to \infty}\left(\frac{800}{r} + 4\pi r^2\right) \;=\; \infty.$$

Alternatively: one could check that $C'(r)$ is negative for $0 < r < \left(\frac{100}{\pi}\right)^{1/3}$ and positive for $\left(\frac{100}{\pi}\right)^{1/3} < r$ by plugging test points such as $r = 1$ and $r = 800$ into $C'(r)$; therefore, $\left(\frac{100}{\pi}\right)^{1/3}$ is a local minimum of $C(r)$ by the First Derivative Test, so it must in fact be a global minimum since it is the only critical point of $C(r)$ on $(0, \infty)$.
We conclude that $r = \left(\frac{100}{\pi}\right)^{1/3}$ cm is the radius that minimizes the cost of producing each can.

Answer: radius = $\left(\frac{100}{\pi}\right)^{1/3}$ **cm**

4. [10 points] Let $f(x)$ be the differentiable function defined by

$$f(x) = x^3 + \cos(x^3), \qquad \text{so} \qquad f'(x) = 3x^2\big(1 - \sin(x^3)\big).$$

For each part below, you must use calculus to find and justify your answers. Clearly state your conclusions and show enough evidence to support them. You may use the graphs of sine and cosine given on the front page, if necessary. Recall that $\pi \approx 3.14$.

a. [3 points] The function $f(x)$ has exactly <u>three</u> critical points **in the interval** $(-1, 2)$. Find them. Give exact answers, and *show your work.*

Solution: Critical points of $f(x)$ occur where $f'(x)$ is either zero or does not exist. Since $f(x)$ is differentiable everywhere, we just need to solve $f'(x) = 0$. From the given formula for $f'(x)$, we see that $f'(x) = 0$ when $x = 0$ or $\sin x^3 = 0$. We know the sine function is zero at

$$x \;=\; \ldots \; -\frac{3\pi}{2}, \; \frac{\pi}{2}, \; \frac{5\pi}{2}, \; \ldots$$

but we know there are only <u>three</u> critical points in $(-1, 2)$ and $x = 0$ is one of them, so we must choose the two points above with cube roots in $(-1, 2)$. Since $-\frac{3\pi}{2} < -1$, also $\left(-\frac{3\pi}{2}\right)^{1/3} < -1$, so the other two critical points of $f(x)$ in $(-1, 2)$ must be $\left(\frac{\pi}{2}\right)^{1/3}$ and $\left(\frac{5\pi}{2}\right)^{1/3}$.

Answer: $f(x)$ has critical points at $x =$ $0, \left(\frac{\pi}{2}\right)^{1/3}, \left(\frac{5\pi}{2}\right)^{1/3}$

b. [4 points] Find the x-coordinates of all *local* minima and maxima of $f(x)$ **on the interval** $(-1, 2)$. If there are none of a particular type, write NONE. *Justify your answers.*

Solution: One solution is to notice that since $3x^2 \geq 0$ and $\sin(x^3) \leq 1$ for all x, the derivative of $f(x)$ is never negative, so $f(x)$ is an increasing function which means it has no local extrema.

Alternatively, we can apply the First Derivative Test to the critical points we found in part (a). Using sign logic and the fact that $3x^3 \geq 0$ and $1 - \sin(x^3) \geq 0$ for all x, we have

$f'(x)$: $\quad + \cdot + = + \quad \big|_{0} \quad + \cdot + = + \quad \big|_{\left(\frac{\pi}{2}\right)^{1/3}} \quad + \cdot + = + \quad \big|_{\left(\frac{5\pi}{2}\right)^{1/3}} \quad + \cdot + = +$

By the First Derivative Test, none of the critical points of $f(x)$ in the interval $(-1, 2)$ is a local extremum of $f(x)$.

Answer: Local min(s) at $x =$ **none** and Local max(es) at $x =$ **none**

c. [3 points] Find the x-coordinates of all *global* minima and maxima of $f(x)$ **on the interval** $[-1, 1]$. If there are none of a particular type, write NONE.

Solution: Since $f(x)$ is an increasing function, its global max on $[-1, 1]$ will occur at the right endpoint $x = 1$, while its global min on $[-1, 1]$ will occur at the left endpoint $x = -1$. Alternatively, we can evaluate $f(x)$ at the endpoints $x = \pm 1$ along with the sole critical point of $f(x)$ in $(-1, 1)$, namely $x = 0$:

$$f(-1) = -1 + \cos(-1), \qquad f(0) = 0 + 1 = 1, \qquad f(1) = 1 + \cos(1).$$

Since $\cos(-1) < 1$ and $\cos(1) > -1$, we see that $f(-1) < f(0) < f(1)$, so the global max of $f(x)$ on $[-1, 1]$ occurs at $x = 1$ and the global min at $x = -1$.

Answer: Global min(s) at $x =$ -1 and Global max(es) at $x =$ 1

5. [14 points] Throughout this problem, let $g(x)$ and $h(x)$ be the functions defined by

$$g(x) = 2x^2 + e^{(x^3)} \quad \text{and} \quad h(x) = e^C + \ln(x^k),$$

where C and k are positive constants.

a. [4 points] Compute the derivatives of $g(x)$ and $h(x)$, remembering that C and k are **constants**. *Show your work.*

Solution: Using the differentiation rules and remembering that C and k are constants, we have

$$g'(x) = 4x + 3x^2 e^{(x^3)} \qquad \text{and} \qquad h'(x) = \frac{1}{x^k} \cdot kx^{k-1} = \frac{k}{x}.$$

Answer: $g'(x) =$ $4x + 3x^2 e^{(x^3)}$ **Answer:** $h'(x) =$ k/x

b. [2 points] Find a formula for the linear approximation $L(x)$ of the function $g(x)$ at the point $x = 1$. Your answer should not include the letter g.

Solution: Using $g(1) = 2(1) + e^{1^3} = 2 + e$ and $g'(1) = 4(1) + 3(1)^2 + e^{1^3} = 4 + 3e$, we have

$$\begin{aligned} L(x) &= g(1) + g'(1)(x-1) \\ &= (2+e) + (4+3e)(x-1). \end{aligned}$$

Answer: $L(x) =$ $(2+e) + (4+3e)(x-1)$

c. [4 points] There exist values of the constants C and k for which the piecewise function

$$f(x) = \begin{cases} g(x) & x \le 1 \\ h(x) & x > 1 \end{cases}$$

is continuous and differentiable. Find such values of C and k, and *show all your work.*

Solution: The functions $g(x)$ and $h(x)$ are continuous on their domains no matter what C and k are, so in order for $f(x)$ to be continuous we just need $g(1) = h(1)$. Solving

$$2 + e = g(1) = h(1) = e^C + \ln(1^k) = e^C$$

for C gives us $C = \ln(2+e)$.

Similarly, since $g(x)$ and $h(x)$ are differentiable on their domains, in order for $f(x)$ to be differentiable we just need $g'(1) = h'(1)$. Solving

$$4 + 3e = g'(1) = h'(1) = \frac{k}{1} = k$$

for k gives us $k = 4 + 3e$.

Answer: $C =$ $\ln(2+e)$ **Answer:** $k =$ $4 + 3e$

Problem 5 continues on the next page.

Problem 5 continues from the previous page. Recall that

$$f(x) = \begin{cases} g(x) & x \le 1 \\ h(x) & x > 1 \end{cases}$$

and $L(x)$ is the linear approximation of $g(x)$ at $x = 1$. For part **d.** below, let C and k be the constants that you found in part **c.**, so $f(x)$ is continuous and differentiable.

d. [4 points] You are given that $g''(x) > 0$ on the domain of $g(x)$, while $h''(x) < 0$ on the domain of $h(x)$. Using this, answer the questions below, and *justify each answer with a brief explanation.*

i. Does the function $L(x)$ from part **b.** give an overestimate or underestimate for $g(x)$ near $x = 1$? Circle your answer, and briefly justify it.

UNDERESTIMATE OVERESTIMATE

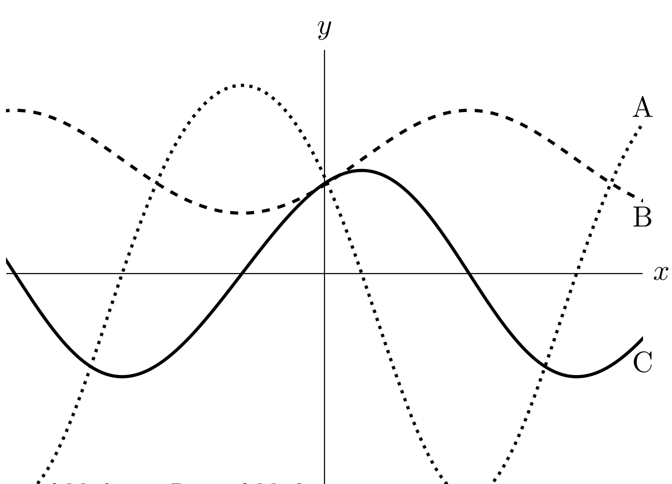
Solution: Since $g''(x) > 0$ on the domain of g, the graph of $g(x)$ is concave up near $x = 1$, so $L(x)$ underestimates $g(x)$ near $x = 1$.

ii. List the x-values of all inflection points of $f(x)$, or write NONE if $f(x)$ has no inflection points. Briefly justify your answer.

Solution: Since $g''(x) > 0$ on the domain of g, we have $f''(x) > 0$ for all $x < 1$. Similarly, since $h''(x) < 0$ on the domain of h, we have $f''(x) < 0$ for all $x > 1$. So $f(x)$ is concave up on $(-\infty, 1)$ and concave down on $(1, \infty)$, which means $f(x)$ has exactly one inflection point, namely at $x = 1$ where its concavity changes.

Answer: $x =$ 1

6. [4 points] Shown below are portions of the graphs of the functions $y = f(x)$, $y = f'(x)$, and $y = f''(x)$. Determine which graph is which, and then, on the answer lines below, indicate after each function the letter A, B, or C that corresponds to its graph. *No work or justification is needed.*



Answer: $f(x)$: B

$f'(x)$: C

$f''(x)$: A

7. [5 points] The equation $x^3 + y^3 - xy^2 = 5$ defines y implicitly as a function of x. Find a formula for $\frac{dy}{dx}$ in terms of x and y. Show every step of your work.

Solution: Implicitly differentiating both sides of $x^3 + y^3 - xy^2 = 5$ with respect to x gives

$$3x^2 + 3y^2\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} \;=\; 0.$$

Rearranging terms and solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx}\Big(3y^2 - 2xy\Big) \;=\; y^2 - 3x^2, \qquad \text{so} \qquad \frac{dy}{dx} \;=\; \frac{y^2 - 3x^2}{3y^2 - 2xy}.$$

Answer: $\frac{dy}{dx} =$ $\frac{y^2 - 3x^2}{3y^2 - 2xy}$

8. [8 points] Let $\mathcal{C}$ be the curve defined by the equation $x^2 + y^3 = 8y$. Note that

$$\frac{dy}{dx} = \frac{2x}{8 - 3y^2}.$$

a. [4 points] Find the coordinates of all points (x, y) on the curve $\mathcal{C}$ where the tangent line to $\mathcal{C}$ is horizontal. Write your answer as a list of points in the form (x, y), or write NONE if there are no such points. *Show all your work.*

Solution: In order to find horizontal tangent lines, we set $\frac{dy}{dx}$ equal to zero and solve:

$$0 \;=\; \frac{dy}{dx} \;=\; \frac{2x}{8 - 3y^2} \qquad \text{when} \qquad x = 0.$$

Now we must find all points on $\mathcal{C}$ such that $x = 0$. So we substitute $x = 0$ into the equation defining $\mathcal{C}$ to obtain $y^3 = 8y$. This equation has three solutions: $y = 0$ and $y = \pm\sqrt{8}$.

Answer: $(0, 0)$, $(0, \sqrt{8})$, $(0, -\sqrt{8})$

b. [4 points] The curve $\mathcal{C}$ intersects the line $y = 1$ at exactly one point with a positive x value. Find an equation of the line tangent to the curve $\mathcal{C}$ at this point. *Show all your work.*

Solution: We plug $y = 1$ into the equation that defines $\mathcal{C}$ to obtain $x^2 + 1 = 8$, or $x^2 = 7$. This has two solutions, $x = \pm\sqrt{7}$, and we want the positive one. So we are looking for an equation of the line tangent to $\mathcal{C}$ at the point $(\sqrt{7}, 1)$. We can find the slope of this line by plugging $x = \sqrt{7}$ and $y = 1$ into the formula given for $\frac{dy}{dx}$ to obtain $\frac{dy}{dx} = \frac{2\sqrt{7}}{8-3(1)^2} = \frac{2\sqrt{7}}{5}$. Thus the tangent line has equation

$$L(x) = 1 + \frac{2\sqrt{7}}{5}\Big(x - \sqrt{7}\Big).$$

Answer: $y =$ $1 + \frac{2\sqrt{7}}{5}\big(x - \sqrt{7}\big)$

9. [11 points] A spherical balloon begins to inflate with air at time $t = 0$, after which time its radius r, volume V, and surface area A increase. Recall that the volume V and surface area A of a sphere of radius r are given by $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$.

a. [5 points] At what rate is air being blown into the balloon when the balloon's radius is 10 cm and its radius is growing at a rate of 2 cm per second? *Include units.*

Solution: We are trying to find $\frac{dV}{dt}$ when $r = 10$ and $\frac{dr}{dt} = 2$. Starting with the formula $V = \frac{4}{3}\pi r^3$, we differentiate with respect to t to obtain

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}. \tag{5}$$

Plugging in $r = 10$ and $\frac{dr}{dt} = 2$ into Equation (3) gives us

$$\frac{dV}{dt} = 4\pi(10^2)(2) = 800\pi \text{ cubic centimeters per second.}$$

Answer: 800π **cm^3/s**

Suppose the volume V and surface area A of the balloon t seconds after it begins to inflate are given by $V = f(t)$ and $A = g(t)$. These functions are invertible, and the function $h(V)$ defined by $h(V) = g\big(f^{-1}(V)\big)$ gives the balloon's surface area as a function of its volume.

b. [3 points] Using the given table of values, find $h'(8)$. Your answer must be a *number*, but need not be simplified.

Solution: Using the Chain Rule and formula for the derivative of an inverse function, we have

$$h'(V) = g'\Big(f^{-1}(V)\Big) \cdot (f^{-1})'(V) = g'\Big(f^{-1}(V)\Big) \cdot \frac{1}{f'\big(f^{-1}(V)\big)}.$$

Since $f(1) = 8$ and thus $f^{-1}(8) = 1$, we therefore have

$$h'(8) = \frac{g'\big(f^{-1}(8)\big)}{f'\big(f^{-1}(8)\big)} = \frac{g'(1)}{f'(1)} = \frac{12.9}{8}.$$

Answer: $h'(8) =$ 12.9/8

t	1	8
r	1.24	2.48
$f(t)$	8	64
$g(t)$	19.34	77.38
dr/dt	0.41	0.10
$f'(t)$	8	8
$g'(t)$	12.90	6.45

c. [3 points] Circle the **one** statement below that is best supported by the equation

$$(h^{-1})'(50) = \frac{1}{4}.$$

i. When the balloon's surface area is 50 cm^2, its volume is 0.25 cm^3.

ii. When the balloon's surface area is 50 cm^2, its surface area is increasing at about $\frac{1}{4}$ the rate at which its volume is increasing.

iii. When the balloon's surface area is 52 cm^2, the balloon is about one cubic centimeter larger in volume than it was when its surface area was 48 cm^2.

iv. The balloon's surface area increases by about 0.25 cm^2 during the time when its volume increases from 50 to 51 cm^3.

v. The balloon's volume increases by about 4 cm^3 during the time when its surface area increases from 50 to 51 cm^2.