Math 115 — Final Exam — April 25, 2025

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
   No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
9. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
10. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	10	
3	13	
4	11	
5	10	

Problem	Points	Score
6	7	
7	9	
8	9	
9	9	
10	10	
Total	100	

1. [12 points] Given below is a table of values for an **even** function $g(x)$. Assume the function $g(x)$ and its derivative $g'(x)$ are defined and continuous on $(-\infty, \infty)$.

x	-2	0	2	4	6	8	10	12
$g(x)$	2	0	2	5	8	3	2	3

Assume that between consecutive values of x given in the table above, $g(x)$ is either **always increasing** or **always decreasing**.

a. [2 points] Find $\displaystyle\int_2^4 \big(2g'(x) - 3x\big)\,dx$.

Solution:

$$\int_2^4 \big(2g'(x) - 3x\big)\,dx \;=\; 2\int_2^4 g'(x), dx - \int_2^4 3x\,dx \;=\; 2\big(g(4) - g(2)\big) - \Big(\frac{3}{2}\cdot 4^2 - \frac{3}{2}\cdot 2^2\Big)$$
$$=\; 2(5-2) - \frac{3}{2}(16-4) \;=\; 6 - 18 \;=\; -12.$$

Answer: -12

b. [3 points] Find the average of value of $g(x)$ on the interval $[-5, 5]$ given that $\displaystyle\int_0^5 4g(x)\,dx = 60$.

Solution: Given that $g(x)$ is even and $\int_0^5 4g(x)\,dx = 60$, the average value of $g(x)$ on $[-5, 5]$ is

$$\frac{1}{5-(-5)}\int_{-5}^5 g(x)\,dx \;=\; \frac{2}{10}\int_0^5 g(x)\,dx \;=\; \frac{2}{40}\int_0^5 4g(x)\,dx \;=\; \frac{1}{20}\cdot 60 \;=\; 3.$$

Answer: 3

c. [2 points] Find a number M that makes the following statement a correct conclusion of the Mean Value Theorem: *There is a number c between 6 and 8 such that $g'(c) = M$.*

Solution: $M = \frac{g(8)-g(6)}{8-6} = \frac{3-8}{2} = -\frac{5}{2}$.

Answer: $M =$ $-5/2$

d. [2 points] Use a right-hand Riemann Sum with 3 equal subdivisions to estimate $\displaystyle\int_0^6 g(x)\,dx$.

Solution: $2\big(g(2) + g(4) + g(6)\big) = 2\big(2 + 5 + 8\big) = 2(15) = 30.$

e. [1 point] Is the estimate in part **d.** an overestimate or an underestimate? Circle your answer below, or circle NEI if there is not enough information to tell.

UNDERESTIMATE [✓ OVERESTIMATE] NEI

Solution: It is an overestimate since $g(x)$ is increasing on $[0, 6]$.

f. [2 points] How many equal subdivisions of $[0, 6]$ are needed so that the difference between the left-hand and right-hand Riemann sum approximations of $\displaystyle\int_0^6 g(x)\,dx$ is exactly 1?

Solution: If we divide $[0, 6]$ into n subintervals, then $\Delta x = \frac{6-0}{n}$, and the error is $|g(6)-g(0)|\frac{6}{n} = 1$. Solving for n gives $n = 6|g(6) - g(0)| = 6\cdot 8 = 48$.

Answer: $n =$ 48

2. [10 points] Consider the family of functions $f(x) = x^2e^{ax}$ where $a > 0$. *Show all your work in each part below.*

a. [2 points] Find the unique value of a such that $f(2) = 12$.

Solution: Plugging in $x = 2$, we have

$$12 = f(2) = 2^2e^{a\cdot 2} = 4e^{2a}, \qquad \text{so} \qquad 3 = e^{2a}.$$

Solving this for a gives us $\ln 3 = 2a$, or $a = \frac{1}{2}\ln 3$.

Answer: $a =$ $\frac{1}{2}\ln 3$

Note: in the parts below, remember that a is a parameter, not the value you just found in part **a**.

b. [2 points] Find the derivative $f'(x)$ in terms of the parameter a.

Solution: Using the Product Rule and keeping in mind that a is a parameter, we have

$$f'(x) = 2xe^{ax} + ax^2e^{ax}.$$

Answer: $f'(x) =$ $2xe^{ax} + ax^2e^{ax}$

c. [2 points] Find all critical points of $f(x)$ in terms of the parameter a.

Solution: Factoring our expression for $f'(x)$ above, we get

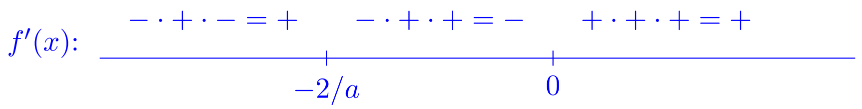
$$f'(x) = 2xe^{ax} + ax^2e^{ax} = xe^{ax}\Big(2 + ax\Big).$$

From this we see that $f'(x)$ is defined everywhere, and equals zero when $x = 0$ or $2 + ax = 0$, that is, when $x = 0$ or $x = -\frac{2}{a}$. (Since we know $a > 0$, we do not need to worry about dividing by a.)

Answer: $x =$ $x = 0$ **and** $x = -2/a$

d. [4 points] Find all local extrema of $f(x)$ in terms of a. If there are none of a particular type, write NONE. *Use calculus to find your answers, and show enough evidence to justify them.*

Solution: We apply the First Derivative Test to $f(x)$. From **c**., we have $f'(x) = xe^{ax}(2 + ax)$. Applying sign logic to these three factors, we get:

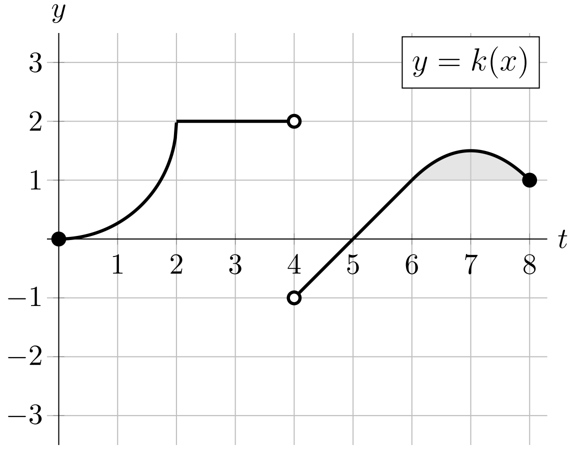


So f' is positive for $x < -2/a$ and $x > 0$, and negative for $-2/a < x < 0$. This means $x = -2/a$ is a local max of $f(x)$, while $x = 0$ is a local min of $f(x)$, by the First Derivative Test.

Answer: Local min(s) at $x =$ 0 and Local max(es) at $x =$ $-2/a$

3. [13 points] A portion of the graph of the function $k(x)$ is shown to the right. Note the following facts about $k(x)$:

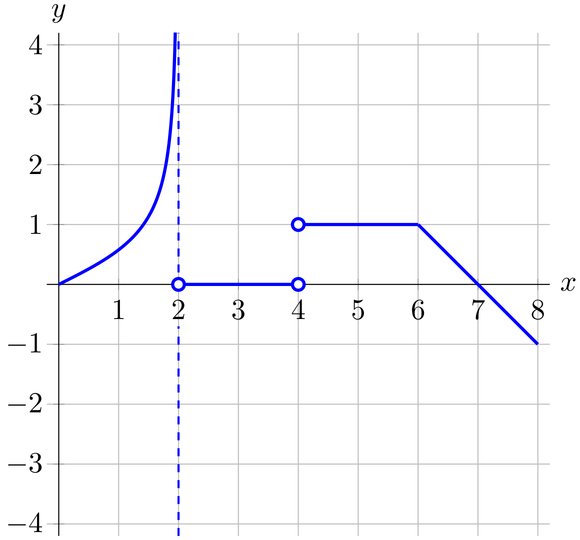
- On the interval $0 \le x \le 2$, the graph of $k(x)$ is a quarter circle.
- On the interval $2 \le x < 4$ and $4 < x \le 6$, $k(x)$ is linear.
- On the interval $6 < x \le 8$, $k(x)$ is quadratic, given by $k(x) = -\frac{1}{2}x^2 + 7x - 23$.
- The shaded region has area $2/3$.



a. [6 points]

On the axes to the right, sketch a detailed graph of $k'(x)$, the derivative of $k(x)$, for $0 < x < 8$. Make sure the following are clear from your graph:

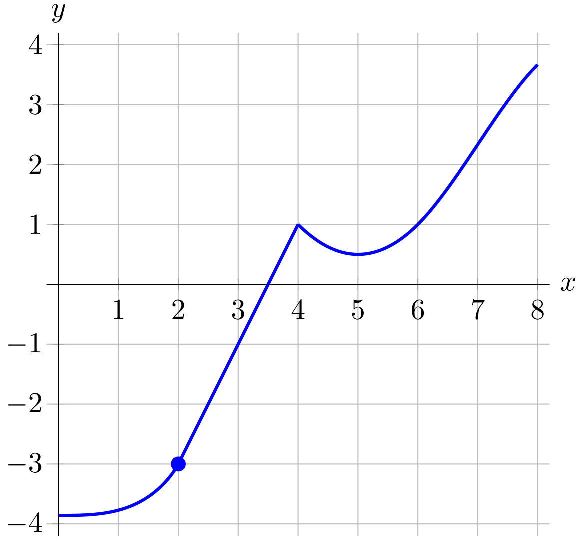
- where $k'(x)$ is undefined;
- any vertical asymptotes of $k'(x)$;
- where $k'(x)$ is zero, positive, or negative;
- where $k'(x)$ is increasing, decreasing, or constant;
- where $k'(x)$ is linear (with correct slope).



b. [7 points]

Let $K(x)$ be a continuous antiderivative of $k(x)$ satisfying $K(2) = -3$. On the axes to the right, sketch a detailed graph of $K(x)$ for $0 \le x \le 8$. Make sure the following are clear from your graph:

- where $K(x)$ is and is not differentiable;
- the approximate values of $K(x)$ at $x = 0$, 2, 3, 4, 6, 7, and 8;
- where $K(x)$ is increasing, decreasing, or constant;
- the concavity and any inflection points of $K(x)$.



4. [11 points] Suppose the rate at which the amount of carbon dioxide (CO_2) in Walden Pond is changing t hours after 6am, in kilograms per hour, is given by the continuous function $h(t)$. Some values of $h(t)$ are given in the table below. Assume that between consecutive values of t given in the table, $h(t)$ is either **always increasing** or **always decreasing**.

t	0	3	6	12	15	18	21
$h(t)$	-2	-5	0	6	8	7	0

No justification is required in any part of this problem, but partial credit may be awarded for work.

a. [2 points] Write an expression involving an integral that represents the change in the amount of CO_2 in Walden pond between 9am and 12 noon.

Answer: $\int_3^6 h(t)\,dt$ kg

b. [2 points] Write an expression involving an integral that represents the average rate of change of CO_2 in Walden pond between 6am and 6pm.

Answer: $\frac{1}{12}\int_0^{12} h(t)\,dt$ kg/hr

c. [7 points] Suppose $H(t)$ is the amount of CO_2 in Walden Pond t hours after 6am, in kilograms, and assume $H(0) = 600$.

i. Put the following quantities in order from *least* to *greatest*.

$H(0)$ $H(3)$ $H(18)$ $H(21)$ $H'(6)$ $h(0)$

Solution: Since $-5 \le h(t) \le -2$ for all $0 \le t \le 3$, we have $600 + 3(-5) \le H(3) < H(0)$. Since $H'(6) = h(6) = 0$, this means $h(0) < H'(6) < H(3) < H(0)$. Also, we have $H(18) < H(21)$ since $h(t) \ge 0$ for all $18 \le t \le 21$. Finally, since $h(t) \ge -5$ for all $0 \le t \le 6$, while $h(t) \ge 0$ for all $6 \le t \le 12$ and $h(t) \ge 6$ for all $12 \le t \le 18$, we have

$$H(18) - H(0) = \int_0^{18} h(t)\,dt \ge 0.$$

Putting all this together gives us $h(0) < H'(6) < H(3) < H(0) < H(18) < H(21)$.

Answer: $h(0)$ < $H'(6)$ < $H(3)$ < $H(0)$ < $H(18)$ < $H(21)$

LEAST GREATEST

ii. Write an expression which does not include a capital "H" that is equal to $H(24)$. You may use the function $h(t)$, along with any integrals, derivatives, or numbers that you want.

Answer: $H(24) = 600 + \int_0^{24} h(t)\,dt$

5. [10 points] Ivan is walking back and forth along a straight line represented by the x-axis, and his position in meters along this path t **seconds** after 12 noon is given by $x = f(t)$. Suppose $f(0) = 0$, so Ivan is $f(t)$ meters east of his starting point t seconds after noon, for all $0 \leq t \leq 100$. Assume Ivan starts out walking eastward, with positive velocity, but at 12:01 is west of his starting point.

Match each expression on the left with the one letter (a) – (h) that it represents, or else write "x" if does not represent any of (a) – (h). Assume all units in (a) – (h) match those given in the introduction above, i.e., *meters* or *meters per second*, as appropriate.

Note: any particular letter (a) – (h) may appear once, more than once, or not at all.

i. __h__ $|f(60)|$

ii. __e__ $f'(60)$

iii. __f__ $|f'(60)|$

iv. __c__ $\dfrac{f(60)-f(0)}{60-0}$

v. __x__ $\left|\dfrac{f(60)-f(0)}{60-0}\right|$

vi. __x__ $\displaystyle\int_0^{60} f(t)\,dt$

vii. __a__ $\displaystyle\int_0^{60} f'(t)\,dt$

viii. __b__ $\displaystyle\int_0^{60} |f'(t)|\,dt$

ix. __c__ $\displaystyle\frac{1}{60-0}\int_0^{60} f'(t)\,dt$

x. __e__ $\displaystyle\lim_{h\to 0}\frac{f(60+h)-f(60)}{h}$

(a) Ivan's net change in position between 12:00 and 12:01.

(b) The total distance Ivan travels between 12:00 and 12:01.

(c) Ivan's average velocity between 12:00 and 12:01.

(d) Ivan's average speed between 12:00 and 12:01.

(e) Ivan's instantaneous velocity at 12:01.

(f) Ivan's instantaneous speed at 12:01.

(g) The furthest distance Ivan gets from his starting point between 12:00 and 12:01.

(h) The distance from Ivan's starting point to his position at 12:01.

(x) NONE OF (a) – (h).

6. [7 points] Continue to assume the setup of the previous problem, so Ivan is $x = f(t)$ meters east of his starting point t seconds after 12 noon, walking back and forth along a straight line. Suppose also that Opal is driving in circles around Ivan and blasting her car stereo, so that:

- the distance r, in meters, between Opal and Ivan t seconds after 12 noon is given by the function $r = g(t)$;
- when Ivan is r meters from Opal, the loudness of Opal's stereo in decibels as perceived by Ivan is given by $L(r) = 100 - 20\log(r)$. [Recall that "log" means log base 10.]

a. [2 points] Find $L'(10)$.

Solution: Using the rule for differentiating logarithmic functions, we have

$$L'(r) = -20 \cdot \frac{1}{(\ln 10)r},$$

so

$$L'(10) = \frac{-2}{\ln 10}.$$

Answer: $L'(10) =$ $-2/\ln(10)$

b. [5 points] At what rate is the loudness of Opal's stereo, as perceived by Ivan, changing with respect to time when Ivan is 10 meters from Opal and moving away from her at a speed of 2 meters per second? *Include units.*

Solution: The loudness of Opal's stereo in decibels, as perceived by Ivan t seconds after noon, is given by the composition $L(g(t))$. To find the rate at which this is changing at a given time t, we differentiate using the Chain Rule to obtain $L'(g(t))g'(t)$. When Ivan and Opal are 10 meters apart and moving away from each other at a speed of 2 meters per second, we have $g(t) = 10$ and $g'(t) = 2$, so at this moment

$$L'(g(t))g'(t) = L'(10) \cdot 2 = -4/\ln(10) \quad \text{decibels per second.}$$

Solution: Alternatively, we can solve this using Leibniz notation. Let us write $y = L(r)$ for the loudness of Opal's stereo, as perceived by Ivan, when they are r meters away from each other. We want to find $\frac{dy}{dt}$ when $r = 10$ and $\frac{dr}{dt} = 2$. Using the Chain Rule, we have

$$\frac{dy}{dt} = \frac{dy}{dr} \cdot \frac{dr}{dt},$$

so when $r = 10$ and $\frac{dr}{dt} = 2$ we have

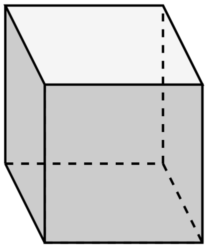
$$\frac{dy}{dt} = \frac{-2}{\ln 10} \cdot 2 = -4/\ln(10) \quad \text{decibels per second,}$$

since we know from part **a.** that $\left.\frac{dy}{dr}\right|_{r=10} = L'(10) = -2/\ln(10)$.

Answer: $-4/\ln(10)$ **decibels per second**

7. [9 points] Suppose 16 square feet of material is available to make a box with a square base and an open top. Find the side length of the base that maximizes the volume of the box.

Show all your work, include units, and fully justify using calculus that you have in fact found side length that maximizes volume.



Solution: Let us write x for the side length of the base, and h for the height of the box. Then the volume and surface area of the box are

$$V = x^2 h \qquad \text{and} \qquad A = x^2 + 4xh,$$

respectively. So we want to maximize V subject to the constraint that $A = 16$. To do this, we need to write V as a function of one variable, so we use the constraint equation to solve for h in terms of x:

$$16 = x^2 + 4xh, \qquad \text{so} \qquad 16 - x^2 = 4xh, \qquad \text{so} \qquad \frac{16 - x^2}{4x} = h.$$

Plugging $h = \frac{16-x^2}{4x}$ into our expression for V, we get

$$V \; = \; V(x) \; = \; x^2 \left(\frac{16 - x^2}{4x} \right) \; = \; 4x - \frac{x^3}{4}.$$

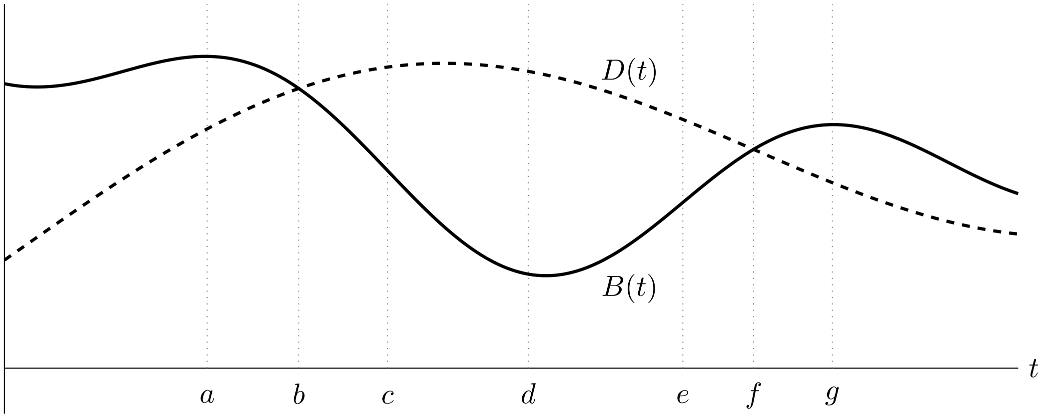
We want to maximize $V(x)$ over the interval $0 < x < 4$, since $x \leq 0$ and $x \geq 4$ would not make sense in the context of the problem. Differentiating to find critical points, we have

$$V'(x) \; = \; 4 - \frac{3}{4}x^2.$$

Setting this equal to zero and solving, we find that $V(x)$ has critical points at $\pm\sqrt{16/3}$. The negative value is outside our domain, so the only critical point in our domain is $x = \sqrt{16/3}$. To check that this is indeed a maximum, we could apply the First or Second Derivative Test, or we could check the behavior of $V(x)$ at the endpoints $x = 0$ and $x = 4$. Plugging 0 and 4 into V, we find that $V(0) = V(4) = 0 < V\left(\sqrt{16/3}\right) = \frac{8}{3}\sqrt{16/3}$, confirming that $x = \sqrt{16/3}$ really is the side length, in feet, that maximizes the volume of the box.

Answer: side length of base = $\sqrt{16/3}$ **feet**

8. [9 points] Shown below are graphs of the birth rate $B(t)$ and death rate $D(t)$ of Antarctic krill in the Southern Ocean over a certain time period, in millions of krill per day. Assume that the *only* changes to the krill population in the Southern Ocean over this time result from births or deaths.



a. [6 points] Seven points in time are labeled on the graph, with the y-axis corresponding to time $t = 0$. In i.–v., write the letter of the <u>one</u> time of these seven that *best* answers the question.

i. At which of the seven times was the krill population largest? b

ii. At which of the seven times was the krill population smallest? f

iii. At which of the seven times was the krill **birth rate** increasing most rapidly? e

iv. At which of the seven times was the krill **population** decreasing most rapidly? d

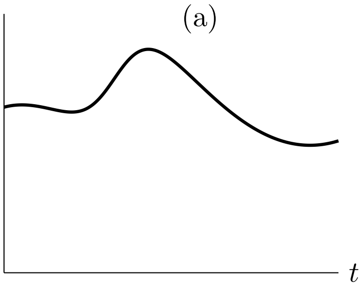
v. At which of the seven times was the krill population closest to what it was at $t = 0$? d

vi. Over which of the following time intervals was the krill population <u>increasing</u>? *Circle all correct answers.*

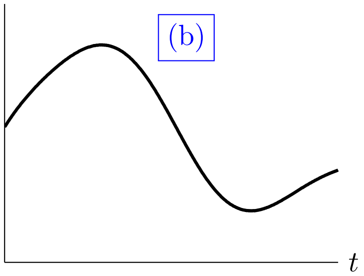
$\boxed{(a, b)}$ (b, c) (c, d) (d, e) (e, f) $\boxed{(f, g)}$ NONE OF THESE

b. [3 points] Which graph below could represent the **total** krill population in the Southern Ocean over the same time period displayed above? *Circle the letter of the <u>one</u> best answer.*

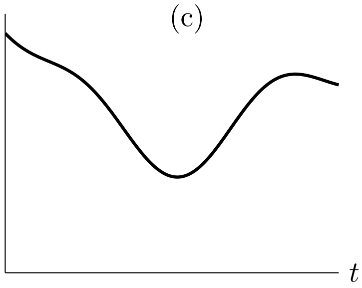
(a)

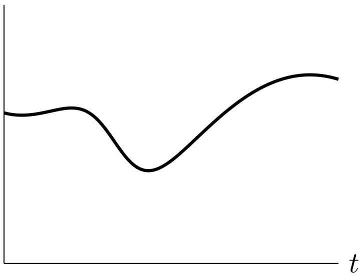


$\boxed{\text{(b)}}$

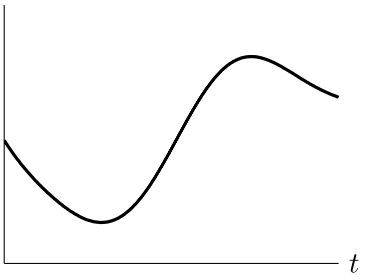


(c)

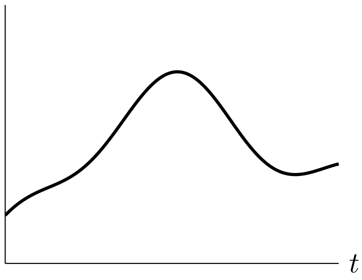




(d)



(e)



(f)

9. [9 points] Roger owns a second farm of 500 trees which produces both timber and wild honey. He plans to harvest q trees for timber in the month of May, while using the remaining $500 - q$ trees for honey production. The cost in dollars for Roger to run his farm in May if he harvests q trees for timber is given by

$$C(q) = 100 + 5q + \frac{q^2}{5}.$$

a. [2 points] During May, each tree cut for timber generates \$85 in revenue, while each tree used for honey production yields \$20. Write an expression for the revenue $R(q)$ that Roger earns in May for harvesting q trees while using the remaining $500 - q$ trees for honey production.

Solution: During May, Roger earns $85q$ dollars from harvesting q trees, and $20(500-q)$ dollars from producing honey from the remaining $500 - q$ trees, so his total revenue is

$$85q + 20(500 - q) \;=\; 65q + 10{,}000 \quad \text{dollars.}$$

Answer: $R(q) =$ $65q + 10{,}000$ dollars

b. [2 points] Find the marginal revenue $\mathrm{MR}(q)$ and the marginal cost $\mathrm{MC}(q)$ of Roger's operation in the month of May.

Solution: Since marginal revenue and marginal costs are the derivatives of revenue and cost, respectively, we get

$$\mathrm{MR}(q) \;=\; R'(q) \;=\; 85 \qquad \text{and} \qquad \mathrm{MC}(q) \;=\; C'(q) \;=\; 5 + \frac{2}{5}q.$$

$\mathrm{MR}(q) =$ 65, and $\mathrm{MC}(q) =$ $5 + \frac{2}{5}q$

c. [3 points] How many trees should Roger allocate to timber production in order to maximize his May profits? *Use calculus, and show your work. You do not need to fully justify your answer, but partial credit may be awarded for work shown.*

Solution: The derivative of profit is $\mathrm{MR}(q) - \mathrm{MC}(q) = 65 - \left(5 + \frac{2}{5}q\right) = 60 - \frac{2}{5}q$, which equals zero when $q = 150$. To check that this sole critical point $q = 150$ of the profit function is indeed a max, we can apply the First Derivative Test and note that $60 - \frac{2}{5}q > 0$ when $0 \le q < 150$ whle $60 - \frac{2}{5}q < 0$ when $150 < q \le 500$. Alternatively, we could apply the Second Derivative Test and note that $-\frac{2}{5} < 0$, so the profit function is concave down near $q = 150$, hence $q = 150$ is a max.

Answer: $q =$ 150 trees

d. [2 points] Suddenly Roger remembers his sustainability pledge to replant exactly as many trees as he cuts down. If the cost of replanting a single harvested tree is \$$b$, find the *smallest* value of b for which dedicating the entire farm to honey production is at least as profitable as producing both timber and honey.

Solution: If Roger must spend \$$b$ to replant every tree he harvests, then his new marginal cost function is $\mathrm{MC}(q) = 5 = \frac{2}{5}q + b$. Marginal revenue is still 65, and we saw in part **c**. that profit is maximized when $\mathrm{MR} = \mathrm{MC}$, that is, when

$$5 + \frac{2}{5}q + b = 65, \qquad \text{or} \qquad \frac{2}{5}q + b = 60.$$

Dedicating the entire farm to honey production means $q = 0$, so plugging $q = 0$ into the equation above and solving for b gives us $b = 60$.

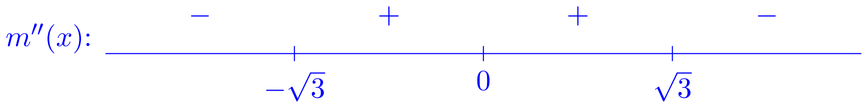
Answer: $b =$ 60

10. [10 points] Let $m(x)$ be a twice-differentiable function that is defined for all real numbers. Suppose the *only* critical point of $m(x)$ is $x = 0$, and that

$$m''(x) = \frac{x^2(9 - x^4)}{(x^4 + 2)^3}.$$

a. [4 points] Find the intervals of concavity of $m(x)$. That is, find the largest open intervals on which $m(x)$ is concave up, and the largest open intervals on which $m(x)$ is concave down. *Show enough work to fully justify your conclusions.*

Solution: We know that $m(x)$ is concave up on intervals where $m''(x) > 0$ and concave down on intervals where $m''(x) < 0$. From the given formula for $m''(x)$, we see that $m''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, so we need to find the sign of $m''(x)$ on the intervals determined by these three points. Notice that $(x^4 + 2)^3$ and x^2 are always nonnegative, so the sign of $m''(x)$ will be the same as the sign of $9 - x^4 = (3 + x^2)(3 - x^2)$. The term $3 + x^2$ is always positive, while $3 - x^2$ is postive only when $-\sqrt{3} < x < \sqrt{3}$. Thus we have the following signs for $m''(x)$:



This means $m(x)$ is concave down on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$, and concave up on $(-\sqrt{3}, 0)$ and $(0, \sqrt{3})$. In fact, since $m''(x) \geq 0$ for all $\sqrt{3} < x < \sqrt{3}$, we see that $m'(x)$ is increasing on the entire interval $(-\sqrt{3}, \sqrt{3})$, so $m(x)$ is concave up on the entire interval $(-\sqrt{3}, \sqrt{3})$.

Answer: Intervals on which $m(x)$ is **concave up**: $(-\sqrt{3}, \sqrt{3})$

Answer: Intervals on which $m(x)$ is **concave down**: $(-\infty, -\sqrt{3})$ **and** $(\sqrt{3}, \infty)$

b. [1 point] Using your work in part (a), list all inflection points of $m(x)$, separated by commas. *No additional justification necessary.*

Answer: $m(x)$ has inflection points at $x =$ $-\sqrt{3}$ **and** $\sqrt{3}$

c. [3 points] Using your work above, classify $x = 0$ as a LOCAL MAX of $m(x)$, a LOCAL MIN of $m(x)$, or NEITHER by circling your answer below, or else circle NEI if there is not enough information to tell. *Include a brief justification of your answer.*

LOCAL MAX [✓ LOCAL MIN] NEITHER NEI

Solution: Since $m(x)$ is differentiable everywhere and has a critical point at $x = 0$, we know $m'(0) = 0$, so $m(x)$ has a horizontal tangent line at $x = 0$. Since we also know $m(x)$ is concave up on $(-\sqrt{3}, \sqrt{3})$, it follows that $m(x)$ has a LOCAL MIN at $x = 0$.

d. [2 points] Find the following limits. Write DNE for any limit that does not exist, even if the limit tends to $\pm\infty$.

i. $\lim\limits_{x \to \infty} m''(x)$

Answer: 0

ii. $\lim\limits_{x \to 0} \frac{m''(x)}{x^2}$

Answer: 9/8