Math 115 — Midterm Exam — February 25, 2026

EXAM SOLUTIONS

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 7 pages including this cover.
3. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	10	
3	10	

Problem	Points	Score
4	12	
5	9	
6	11	
Total	60	

1. [8 points] Katie is biking on a long, straight path. Her distance, in kilometers (km), from the start of the path t minutes after she begins is given by the function $K(t)$. Some values of $K'(t)$, the **derivative** of $K(t)$, are given in the table below.

t	0	5	10	15	20
$K'(t)$	0	0.27	0.37	0.45	0.5

a. [3 points] Estimate $K''(8)$. Show your work and **include units**.

Solution: We approximate $K''(8)$, the derivative of K' at $t = 8$, by computing the average rate of change of $K'(t)$ over the interval $5 \le t \le 10$, since this is the smallest interval in the table that includes $t = 8$.

$$K''(8) \approx \frac{K'(10) - K'(5)}{10 - 5} = \frac{0.37 - 0.27}{10 - 5} = \frac{0.10}{5} = 0.02.$$

Since the units for the numerator and denominator are kilometers per minute and minutes, respectively, the units on our estimate are km/min^2.

Answer: $K''(8) \approx$ 0.02 **km/min**^{2}

b. [1 point] Which **one** of the following statements best describes the first 20 minutes of Katie's ride? Circle the numeral of your answer.

i. $\boxed{\text{Katie started out slowly, but then began to speed up.}}$

ii. Katie started out fast, but then began to slow down as she got tired.

iii. Katie rode at a steady speed for this part of her ride.

c. [4 points] After 20 minutes, Katie has ridden for 7 km, and still has 2 km to go. Find a formula for $L(t)$, the linear approximation to $K(t)$ at $t = 20$, and use it to estimate the total amount of time Katie's bike ride will take.

Solution: The linear approximation $L(t)$ of the function $K(t)$ at $t = 20$ is

$$L(t) = K(20) + K'(20)(t - 20) = 7 + 0.5(t - 20).$$

To estimate the total amount of time Katie's bike ride will take using $L(t)$, we solve the equation $L(t) = 7 + 2 = 9$ for t:

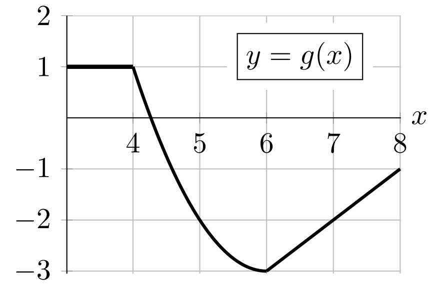
$$\begin{aligned} 7 + 0.5(t - 20) &= 9 \\ 0.5(t - 20) &= 2 \\ t - 20 &= 4 \\ t &= 24. \end{aligned}$$

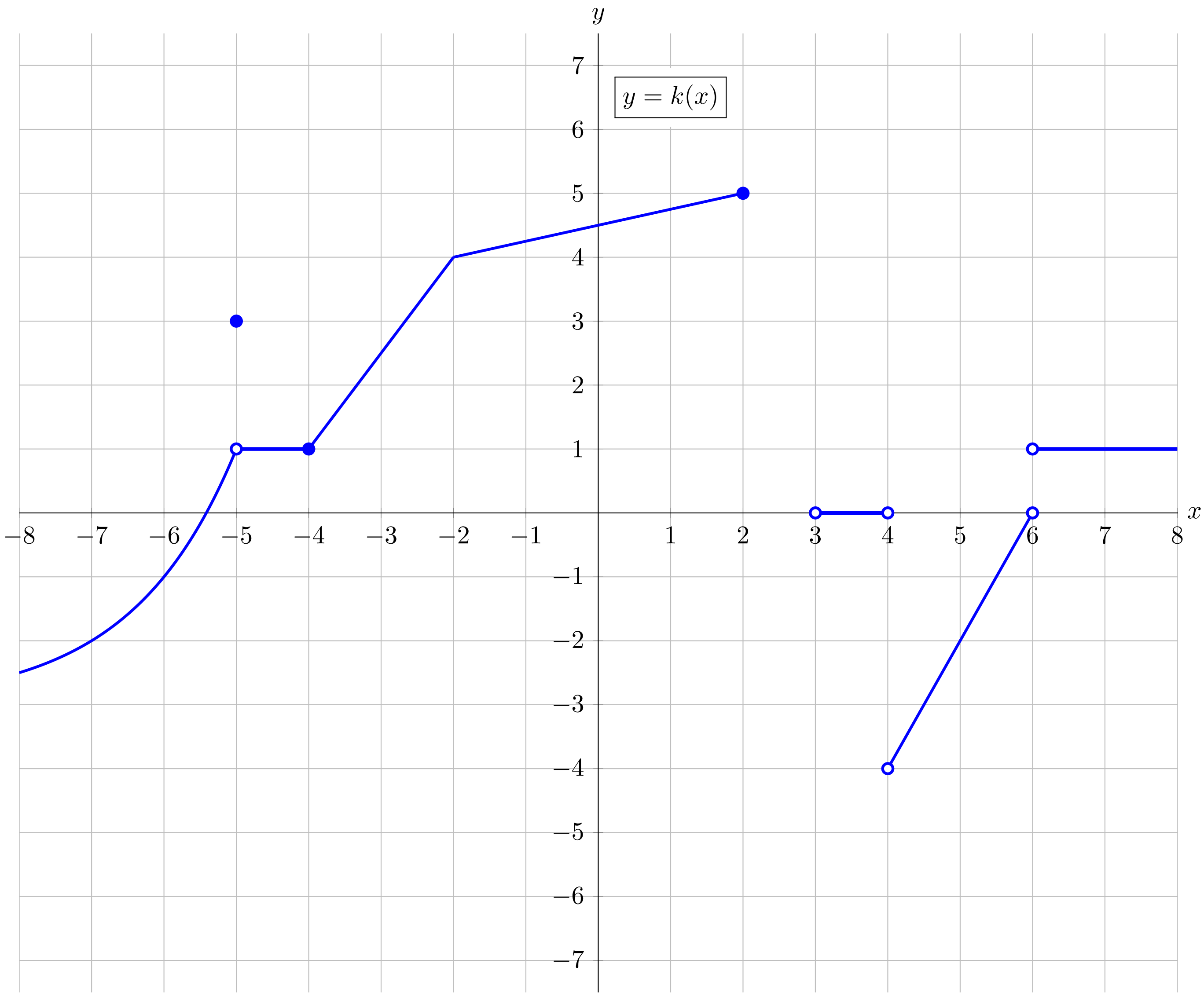
$L(t) =$ $7 + \frac{1}{2}(t - 20)$

length of Katie's ride $\approx$ 24 minutes

2. [10 points] On the axes provided below, sketch the graph of a single function $k(x)$ that satisfies all of the following conditions.

- $k'(x)$ and $k''(x)$ exist and are positive on $-8 < x < -5$.
- $\lim_{x \to -5} k(x) = 1$.
- $k(x)$ is defined at $x = -5$ but is <u>not</u> continuous there.
- The average rate of change of $k(x)$ from -4 to 2 is $\frac{2}{3}$.
- $k(x)$ is defined and continuous on the interval $-4 \leq x \leq 2$, but does <u>not</u> satisfy the hypotheses of the Mean Value Theorem on this interval.
- On the interval $3 < x < 8$, the function $k(x)$ is the derivative of the function $g(x)$, which is shown in the graph to the right. Note that the graph of $g(x)$ has corners at $x = 4$ and $x = 6$.





3. [10 points] The number of years, t, that it takes a quantity of money to double when it earns an annual interest rate of $r\%$ is given by

$$t = f(r) = \frac{\ln(2)}{\ln\left(1 + \frac{r}{100}\right)}.$$

a. [3 points] Find a formula for $f^{-1}(t)$; that is, solve the above equation for r in terms of t. *Show your work.*

Solution: First, multiply both sides by $\ln\left(1 + \frac{r}{100}\right)$ and divide both sides by t to get

$$\ln\left(1 + \frac{r}{100}\right) = \frac{\ln(2)}{t}.$$

Then apply the exponential function with base e to both sides to get

$$1 + \frac{r}{100} = e^{(\ln 2)/t} = 2^{1/t}.$$

Finally, subtract 1 from both sides and then multiply by 100 to get

$$r = 100 \cdot \left(e^{(\ln 2)/t} - 1\right) = 100 \cdot \left(2^{1/t} - 1\right).$$

Answer: $r = f^{-1}(t) =$ $100 \cdot \left(e^{(\ln 2)/t} - 1\right)$ **or** $100 \cdot \left(2^{1/t} - 1\right)$

b. [2 points] Write a single equation that represents the statement below. (Your answer may include the letter f.)

Your money takes twice as long to double when it earns 4% interest as it does when it earns 8.2% interest.

Answer: $f(4) = 2f(8.2)$

c. [2 points] Circle the <u>**one**</u> equation below that best represents the following statement:

Your money takes about three months longer to double when it earns 5.2% interest than when it earns 5.3% interest.

i. $f'(5.2) = 0.25$

ii. $f'(5.2) = -30$

iii. $f'(5.3) - f'(5.2) = 3$

iv. $\boxed{f'(5.2) = -2.5}$

v. $f'(5.2) = 3$

vi. $f(5.3) - f(5.2) = 0.25$

d. [3 points] Complete the sentence below by writing positive numbers in the blanks and circling either HIGHER or LOWER to give a practical interpretation of the following equation:

$$(f^{-1})'(12) = -0.5.$$

If your money will double in 12 *years at the current interest rate, but you want it to double six months earlier, then you will need to invest at an interest rate that is about* 0.25 *percent* $\boxed{\textit{higher}}$ */ lower than the current one.*

4. [12 points] The **continuous** function $w(t)$ is given by the piecewise formula below. Also given is a partial formula for its derivative $w'(t)$. In case it is helpful, recall that $e \approx 2.7$.

$$w(t) = \begin{cases} 3e^t & t \le 0 \\ \dfrac{1}{3}t^3 - 2t^2 + 3t + 3 & t > 0 \end{cases} \qquad w'(t) = \begin{cases} 3e^t & t < 0 \\ ?? & t = 0 \\ (t-1)(t-3) & t > 0 \end{cases}$$

For parts **a.**, **c.**, and **d.** below, you must use calculus to find and **justify** your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

a. [2 points] Find $w'(0)$, or write DNE if it does not exist. Show your work or explain.

Solution: Since $w(t)$ is continuous at 0, $w'(0)$ exists if the left and right limits $\lim\limits_{t\to 0^-} w'(t)$ and $\lim\limits_{t\to 0^+} w'(t)$ are the same, in which case $w'(0)$ is their common value. Since

$$\lim_{t\to 0^-} w'(t) \;=\; \lim_{t\to 0^-} 3e^t \;=\; 3e^0 \;=\; 3$$

and

$$\lim_{t\to 0^+} w'(t) \;=\; \lim_{t\to 0^+} (t-1)(t-3) \;=\; (0-1)(0-3) \;=\; 3,$$

we see that $w'(0)$ *does* exist, and equals 3.

Answer: $w'(0) =$ 3

b. [2 points] Find all critical points of $w(t)$, or write NONE if there are none. (*No justification required.*)

Answer: Critical point(s) at $t =$ $1, 3$

c. [4 points] Find the t-coordinates of all global minimum(s) and global maximum(s) of $w(t)$ **on the interval $[-1, 3]$**. If there are none of a particular type, write NONE.

Solution: We just need to evaluate $w(t)$ at its critical points in $[-1, 3]$, and also at the endpoints $t = -1$ and $t = 3$, and determine the greatest and least values from the resulting list. Since

$$w(-1) = 3e^{-1} = \frac{3}{e}, \qquad w(1) = \frac{13}{3}, \qquad \text{and} \qquad w(3) = 3,$$

with $\frac{3}{e} < 3 < \frac{13}{3}$, w has a global min on $[-1, 3]$ at $t = -1$, and a global max on $[-1, 3]$ at $t = 1$.

Answer: Global min(s) at $t =$ -1

Answer: Global max(es) at $t =$ 1

d. [4 points] Find the t-coordinates of all global minimum(s) and global maximum(s) of $w(t)$ **on the interval $(-\infty, \infty)$**. If there are none of a particular type, write NONE.

Solution: Since $\lim\limits_{t\to\infty} w(t) = \lim\limits_{t\to\infty} \left(\frac{t^3}{3} - 2t^2 + 3t + 3\right) = \infty$, $w(t)$ has no global max on $(-\infty, \infty)$. Since $\lim\limits_{t\to-\infty} w(t) = \lim\limits_{t\to-\infty} 3e^t = 0$ and 0 is less than the value of $w(t)$ at all of its critical points (namely $w(1) = \frac{13}{3}$ and $w(3) = 3$), we see that $w(t)$ has no global min on $(-\infty, \infty)$ either.

Answer: Global min(s) at $t =$ **NONE**

Answer: Global max(es) at $t =$ **NONE**

5. [9 points] A function $u(x)$ and its first and second derivatives are given by the formulas

$$u(x) = 2x^2 - \ln|x|, \qquad u'(x) = \frac{(2x-1)(2x+1)}{x}, \qquad \text{and} \qquad u''(x) = \frac{1}{x^2} + 4.$$

Note that $u(x)$ is defined for all real numbers **except** $x = 0$.

a. [2 points] List the x-coordinates of all critical points of $u(x)$. (*No justification required.*)

Answer: Critical points at $x =$ $-1/2,\ 1/2$

b. [4 points] **Use the first derivative test** to find the x-coordinates of all local minima and local maxima of $u(x)$. If there are none of a particular type, write NONE. Be sure you show enough evidence to support your conclusions.

Hint: Remember that $u(x)$ is undefined at $x = 0$.

Solution: In order to apply the first derivative test to the critical points $x = \pm\frac{1}{2}$ of $u(x)$, we need to dermine the sign of $u'(x)$ on either side of $\pm\frac{1}{2}$. Using sign logic, we make the following table of signs:

	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < 0$	$0 < x < \frac{1}{2}$	$\frac{1}{2} < x$
$2x-1$	$-$	$-$	$-$	$-$
$2x+1$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$u'(x)$	$\frac{-\cdot-}{-} = -$	$\frac{-\cdot+}{-} = +$	$\frac{-\cdot+}{+} = -$	$\frac{+\cdot+}{+} = +$

Or, using a number line:

$u'(x)$: $\frac{-\cdot-}{-} = -$ | $-\frac{1}{2}$ | $\frac{-\cdot+}{-} = +$ | 0 | $\frac{-\cdot+}{+} = -$ | $\frac{1}{2}$ | $\frac{+\cdot+}{+} = +$

Note that even though $x = 0$ is not a critical point of $u(x)$, we need to include it on our number line since $u'(x)$ changes sign there.

Finally, we interpret the signs: since $u'(x)$ changes from negative to positive at both $-\frac{1}{2}$ and $\frac{1}{2}$, each of these points is a local min of $u(x)$. Since $u(x)$ has no other critical points, it has no local max. (Note that $x = 0$ cannot be a local extremum of $u(x)$, since $u(0)$ is undefined.)

Answer: Local min(s) at $x =$ $-1/2,\ 1/2$ and Local max(es) at $x =$ **NONE**

c. [3 points] Note that $u''(x)$ is positive wherever it is defined.

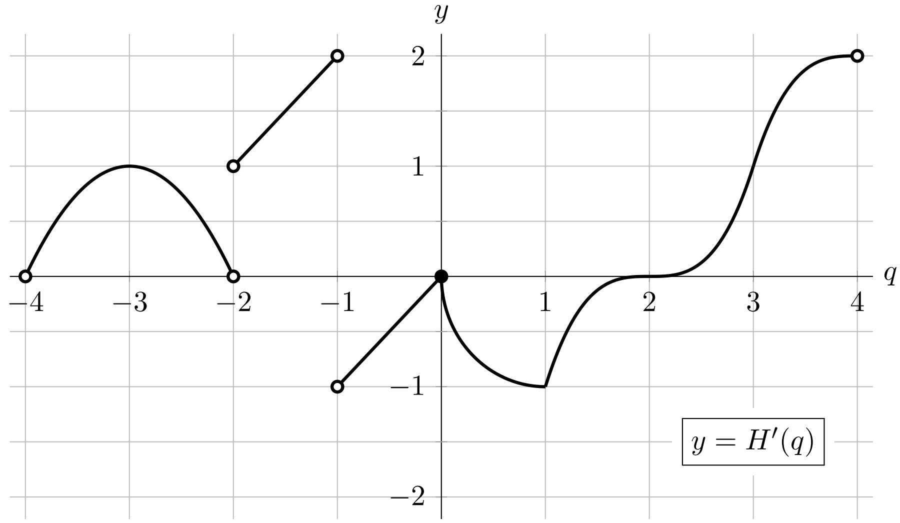
i. How could you use this fact to verify your answer to part **b.**? Briefly explain.

Solution: We could use the Second Derivative Test: since $u''(x)$ is positive near $\pm 1/2$, the graph of $u(x)$ is concave up near $\pm 1/2$, so the critical points $\pm 1/2$ are local minima of $u(x)$.

ii. What does this fact tell you about the number of inflection points of $u(x)$? Briefly explain.

Solution: It tells us that $u(x)$ has no inflection points, since inflection points of $u(x)$ are points at which $u(x)$ changes concavity, or equivalently where $u''(x)$ changes sign.

6. [11 points] Suppose $H(q)$ is a **continuous** function defined everywhere on the interval $(-4, 4)$. A graph of the **derivative** $H'(q)$ is given below.



a. [2 points] Circle all intervals listed below on which $H(q)$ is **increasing**.

$\boxed{(-4,-3)}$ $\boxed{(-3,-2)}$ $(0,1)$ $(1,2)$ $\boxed{(2,3)}$ NONE OF THESE

b. [2 points] Circle all intervals listed below on which $H(q)$ is **concave down**.

$(-4,-3)$ $\boxed{(-3,-2)}$ $\boxed{(0,1)}$ $(1,2)$ $(2,3)$ NONE OF THESE

c. [2 points] Circle all intervals listed below on which $H''(q)$ is **decreasing**.

$\boxed{(-4,-3)}$ $\boxed{(-3,-2)}$ $(0,1)$ $\boxed{(1,2)}$ $(2,3)$ NONE OF THESE

d. [2 points] Circle all points listed below that are **inflection points** of $H(q)$.

$\boxed{q=-3}$ $\boxed{q=-2}$ $\boxed{q=0}$ $q=2$ $q=3$ NONE OF THESE

e. [1 point] Circle all points c below for which the limit $\lim_{q\to c} H(q)$ **does not exist**.

$c=-2$ $c=-1$ $c=0$ $c=1$ $c=2$ $\boxed{\text{NONE OF THESE}}$

f. [1 point] Circle all points c below for which the limit $\lim_{q\to c} H'(q)$ **does not exist**.

$\boxed{c=-2}$ $\boxed{c=-1}$ $c=0$ $c=1$ $c=2$ NONE OF THESE

g. [1 point] Circle all points c below for which the limit $\lim_{q\to c} H''(q)$ **does not exist**.

$\boxed{c=-2}$ $c=-1$ $\boxed{c=0}$ $\boxed{c=1}$ $c=2$ NONE OF THESE