Math 115 — Final Exam — April 27, 2026

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 12 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
   No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
9. Recall that the area A and circumference C of a circle of radius r are $A = \pi r^2$ and $C = 2\pi r$.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.
12. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	13	
3	5	
4	4	
5	4	
6	3	

Problem	Points	Score
7	5	
8	8	
9	9	
10	11	
11	8	
12	9	
Total	90	

1. [11 points]
Some values of the increasing, differentiable function $h(x)$ and its derivative $h'(x)$ are given in the table to the right.

x	-1	1	3	5
$h(x)$	-3	-2	1	3
$h'(x)$	$\frac{1}{2}$	1	2	$\frac{1}{3}$

a. [2 points] Let $W(x) = h\left(\frac{x}{2}\right)\ln(x)$. Find $W'(6)$.

Solution: By the Product Rule and Chain Rule, we have $W'(x) = \frac{1}{2}h'(\frac{x}{2})\ln(x) + h(\frac{x}{2})/x$, so

$$W'(6) \;=\; \frac{1}{2}h'(3)\ln(6) + \frac{h(3)}{6} \;=\; \frac{1}{2}\cdot 2\ln(6) + \frac{1}{6} \;=\; \ln(6) + \frac{1}{6}.$$

Answer: $\ln(6) + \dfrac{1}{6}$

b. [2 points] Find the average value of $h'(x)$ on the interval $[-1, 5]$.

Solution: Using the Net Change Theorem, the average value of $h'(x)$ on the interval $[-1, 5]$ is

$$\frac{1}{5-(-1)}\int_{-1}^{5} h'(x)\,dx \;=\; \frac{1}{6}\Big(h(5) - h(-1)\Big) \;=\; \frac{3-(-3)}{6} \;=\; \frac{6}{6} \;=\; 1.$$

Answer: 1

c. [3 points] Find $\displaystyle\int_1^3 \big(4h''(x) + \cos(x)\big)\,dx$.

Solution: Using the Net Change Theorem, we have

$$\begin{aligned}\int_1^3 \big(4h''(x) + \cos(x)\big)\,dx \;&=\; 4\int_1^3 h''(x)\,dx + \int_1^3 \cos(x)\,dx \;=\; 4\Big(h'(3) - h'(1)\Big) + \Big(\sin(3) - \sin(1)\Big)\\ &=\; 4(2-1) + \sin(3) - \sin(1) \;=\; 4 + \sin(3) - \sin(1).\end{aligned}$$

Answer: $4 + \sin(3) - \sin(1)$

d. [2 points] Use a right-hand Riemann sum with two equal subdivisions to estimate $\displaystyle\int_1^5 h(x)\,dx$. Write out all the terms in your sum, which you do not need to simplify.

Solution: $h(3)\cdot 2 + h(5)\cdot 2 \;=\; 1\cdot 2 + 3\cdot 2 \;=\; 8.$

e. [2 points] How many equal subdivisions of $[1, 5]$ are needed so that the difference between the left and right Riemann sum approximations of $\displaystyle\int_1^5 h(x)\,dx$ is exactly 1?

Solution: If we use n equal subdivisions, then each subinterval has width $\Delta x = \frac{5-1}{n} = \frac{4}{n}$, which makes the error between the left and right Riemann sums

$$\big(h(5) - h(1)\big)\Delta x \;=\; \big(3 - (-2)\big)\frac{4}{n} \;=\; \frac{20}{n}.$$

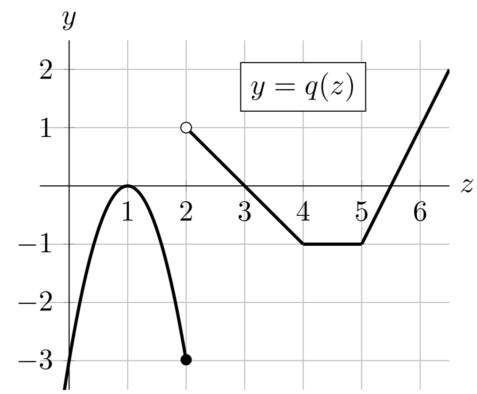
Setting this error equal to 1 and solving for n gives us $n = 20$.

Answer: 20

2. [13 points]

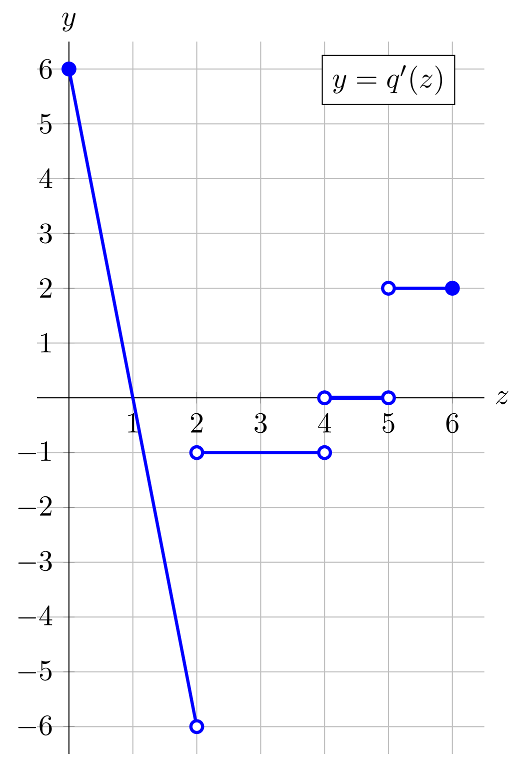
A portion of the graph of the function $q(z)$ is shown to the right. Note the following about $q(z)$:

- on the interval $0 \le z \le 2$, $q(z) = -3(z-1)^2$;
- $q(z)$ is linear on each of the intervals $(2, 4)$, $(4, 5)$, and $(5, 6)$.



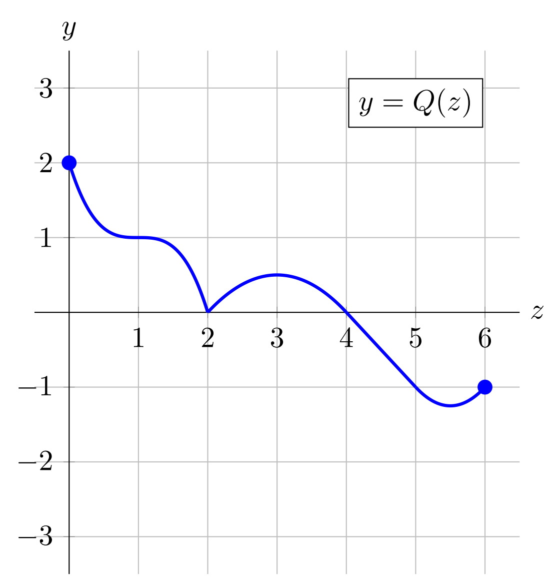
a. [6 points] On the axes to the right, sketch a detailed graph of $q'(z)$, the *derivative* of $q(z)$, for $0 < z < 6$. Make sure the following are clear from your graph:

- where $q'(z)$ is undefined;
- any vertical asymptotes of $q'(z)$;
- where $q'(z)$ is zero, positive, and negative;
- where $q'(z)$ is increasing, decreasing, and constant.



b. [7 points] Let $Q(z)$ be a continuous *antiderivative* of $q(z)$ with $Q(3) = \frac{1}{2}$. On the axes to the right, sketch a detailed graph of $Q(z)$ for $0 \le z \le 6$. Make sure that the following are clear from your graph:

- where $Q(z)$ is and is not differentiable;
- the values of $Q(z)$ at $z = 0, 1, 2, 3, 4, 5, 6$;
- where $Q(z)$ is increasing, decreasing, and constant;
- where $Q(z)$ is linear (with correct slope);
- the concavity of $Q(z)$.



3. [5 points] Let

$$j(x) = \ln(x^2 + 1).$$

Use the limit definition of the derivative to write an explicit expression for $j'(7)$. *Your answer should not involve the letter j. Do not attempt to evaluate or simplify the limit.* Write your final answer in the answer box provided below.

Answer: $j'(7) = \boxed{\lim_{h \to 0} \frac{\ln\left((7+h)^2 + 1\right) - \ln(7^2 + 1)}{h}}$

4. [4 points] Consider the family of circles defined by the equations

$$(x - a)^2 + y^2 = r^2,$$

where a and r are parameters with $r > 0$. Jessie draws a circle from this family that passes through $(0, 10)$ with a slope of $-\frac{1}{2}$. Find the exact value of both parameters a and r for Jessie's circle. *Show your work.*

Solution: Implicitly differentiating both sides of the given equation with respect to x (and remembering to treat a and r as constants), we get

$$2(x - a) + 2y\frac{dy}{dx} = 0.$$

Plugging in $x = 0$, $y = 10$, and $\frac{dy}{dx} = -\frac{1}{2}$, we find

$$2(0 - a) + 2(10)(-1/2) = 0, \qquad \text{or} \qquad -2a - 10 = 0,$$

so $a = -5$.

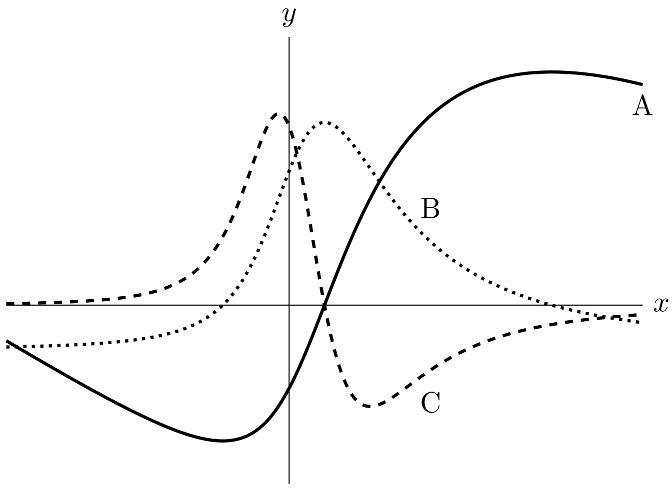
Now we can plug $x = 0$, $y = 10$, and $a = -5$ into the original equation and solve for r to get

$$(0 - (-5))^2 + 10^2 = r^2, \qquad \text{or} \qquad 25 + 100 = r^2,$$

so $r = \sqrt{125}$.

Answer: $a =$ -5 and $r =$ $\sqrt{125}$

5. [4 points] Shown below is a portion of the graph of the function $y = f(x)$, along with portions of the graphs of an *antiderivative* $F(x)$ of $f(x)$, and the *derivative* $f'(x)$ of $f(x)$. Determine which graph is which, and then, on the answer lines below, indicate after each function the letter A, B, or C that corresponds to its graph. No work or justification is needed.



Answer: $F(x)$: A

$f(x)$: B

$f'(x)$: C

6. [3 points] Evaluate the following limits. For any limits that do not exist, including if they are limits that diverge to $\pm\infty$, write DNE. No work or justification is necessary.

$\lim_{x\to\infty} \frac{x(4x^3+1)^2}{5x^7 - x^4 + 12} =$ 16/5 $\qquad \lim_{x\to\infty} \frac{\log(x)}{1-2^x} =$ 0 $\qquad \lim_{x\to\infty} \frac{e^{1/x}+e^{-x}}{2} =$ 1/2

7. [5 points] A company's maximum profit is earned when it produces $q = 6$ units of their product. If the company can produce as many units as it wants and its cost function in thousands of dollars is

$$C(q) = \frac{1}{3}(q-3)^3 + q + 20 \qquad \text{for } q \geq 0,$$

which of the statements i. – v. below could be true? *Circle all answers that could be true*, or else circle "vi." if none of them could be true.

$\boxed{\text{i.}}$ $MR(6) = MC(6)$, where MR and MC are the company's marginal revenue and marginal cost functions, respectively.

ii. The company's fixed cost is 20,000 dollars.

$\boxed{\text{iii.}}$ The company's marginal cost function is $MC(q) = (q-3)^2 + 1$.

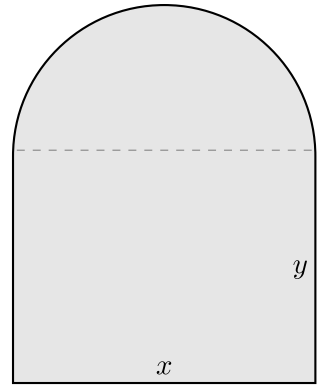
iv. The company's marginal revenue function is $MR(q) = 26$.

$\boxed{\text{v.}}$ The company's revenue function is $R(q) = \begin{cases} 12q & 0 \leq q \leq 6; \\ 18 + 9q & q > 6. \end{cases}$

vi. NONE of the above.

Solution: Note that $MC(q) = C'(q) = (q-3)^2+1$, so iii. is true. Furthermore, when $q = 6$ we have $MC(q) = (6-3)^2 + 1 = 10$, which is enough to show that iv. cannot be true but v. could be true. Indeed, if $R(q)$ is as in v., then $MR(q) - MC(q) = 2 > 0$ for $q < 6$, while $MR(q) - MC(q) = -1 < 0$ for $q > 6$, showing that $R(q) - C(q)$ has a max at $q = 6$.

8. [8 points]



A Norman window has the shape of a semicircle positioned on top of a rectangle, as pictured to the right. An architect is designing a large Norman window with a perimeter of 10 meters. The architect wants the window to let in as much light as possible, so she is trying to determine the dimensions that will maximize the area of the window. Suppose x is the base and y is the height of the rectangular portion of the window in meters, as shown in the figure.

a. [3 points] Find a formula for the area A of the window in terms of x and y.

Solution: The window consists of the rectangle and the semicircle. The rectangle has area xy. Since the semicircle has radius $\frac{x}{2}$, its area is $\frac{1}{2}\cdot\pi\left(\frac{x}{2}\right)^2$. Therefore, the area A of the window is

$$A = xy + \frac{1}{2}\cdot\pi\left(\frac{x}{2}\right)^2 = xy + \frac{\pi}{8}x^2.$$

Answer: $A =$ $xy + \frac{\pi}{8}x^2$

b. [3 points] Find a formula for the area $A(x)$ of the window in terms of x only.

Solution: Since the semicircle has radius $\frac{x}{2}$ meters, the perimeter of the semicircle is $\frac{1}{2}\cdot(2\pi\frac{x}{2}) = \frac{\pi}{2}x$ meters. Therefore, the perimeter of the window is $x + 2y + \frac{\pi}{2}x = 2\left(y + \frac{2+\pi}{4}x\right)$ meters. We solve for y in terms of x using the constraint

$$\text{perimeter of window in meters} = 2\left(y + \frac{2+\pi}{4}x\right) = 10$$
$$y = 5 - \left(\frac{2+\pi}{4}\right)x.$$

Plugging this expression for y into our answer to part a., we get

$$A(x) = x\left(5 - \left(\frac{2+\pi}{4}\right)x\right) + \frac{\pi}{8}x^2.$$

Answer: $A(x) =$ $x\left(5 - \left(\frac{2+\pi}{4}\right)x\right) + \frac{\pi}{8}x^2$

c. [2 points] In the context of this problem, what is the domain of the function $A(x)$ that you found in part **b**?

Solution: In the context of this problem, both x and y should be positive. This means

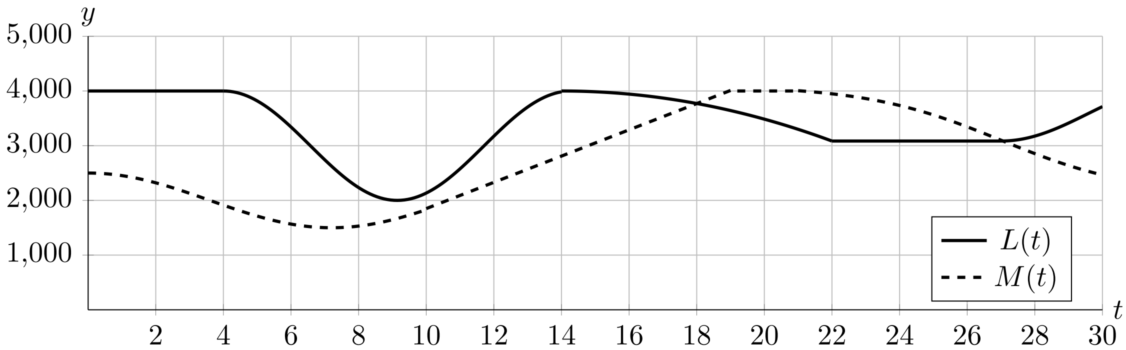
$$0 < y = 5 - \left(\frac{2+\pi}{4}\right), \qquad \text{so} \qquad \left(\frac{2+\pi}{4}\right)x < 5, \qquad \text{so} \qquad x < \frac{20}{2+\pi}.$$

Note: while $x = 0$ does not make sense in the context of this problem, $y = 0$ does still correspond to a reasonable window even if it would arguably not be a Norman window; consequently, $0 < x \le \frac{20}{2+\pi}$ is also an acceptable answer.

Answer: $0 < x < \dfrac{20}{2+\pi}$

9. [9 points] James is running a juice factory, and is examining some production and sales numbers from 2025. In the month of November, the factory produced juice at a rate of $L(t)$ gallons per day, and sold juice at a rate of $M(t)$ gallons per day, t days after midnight on November 1st. Any surplus juice that was produced but not yet sold was stocked in a refrigerator, and the refrigerators contained 1,200 gallons of juice at midnight on November 1.

Graphs of the continuous functions $L(t)$ and $M(t)$ are shown below.



a. [2 points] Write a sentence, including units, that gives a practical interpretation of the equation

$$\int_0^{30} L(t)\,dt = 101{,}726.$$

Solution: The factory produced a total of 101,726 gallons of juice during November 2025.

b. [1 point] At what t value was the stock of refrigerated juice at the factory decreasing the fastest?

Answer: $t =$ 22

c. [1 point] At what t value was the stock of refrigerated juice at the factory greatest?

Answer: $t =$ 18

d. [2 points] Write an expression for the total number of gallons of refrigerated juice in stock at the factory at the end of November.

Answer: $1200 + \int_0^{30} \big(L(t) - M(t)\big)\,dt$

e. [2 points] Write an expression for the *change* in the number of gallons of refrigerated juice in the factory between $t = 4$ and $t = 8$.

Answer: $\int_4^{8} \big(L(t) - M(t)\big)\,dt$

f. [1 point] Given that the average amount of juice sold per month by the factory in 2025 was 97,500 gallons, complete the statement below by writing $>$, $<$, or $=$ in the blank.

The amount of juice sold in Nov. 2025 was $<$ the average amount sold per month in 2025.

10. [11 points] A function $P = f(r)$ and its first and second derivatives are given by the formulas

$$f(r) = \frac{c^2 r}{(r+k)^2} \qquad f'(r) = \frac{c^2(k-r)}{(r+k)^3}, \qquad \text{and} \qquad f''(r) = \frac{2c^2(r-2k)}{(r+k)^4},$$

where c and k are **positive** constants. Note that $f(r)$ is defined for all real numbers *except* $r = -k$. In this problem, you must use calculus to find and **justify** your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

a. [4 points] Find all **local extrema** of $f(r)$, and classify each as a max or a min. If there are none of a particular type, write NONE. Your answer may involve c and/or k.

Solution: Since $f'(r)$ exists everywhere but $r = -k$, and $f(r)$ is undefined at $r = -k$, any local extrema of $f(r)$ must occur at points where $f'(r) = 0$. But $f'(r) = 0$ only when $r = k$. To check whether $r = k$ is indeed a local extremum of $f(r)$, and to determine which type if it is, we use the First Derivative Test.

For $0 < r < k$, we have $c^2 > 0$, $k - r > 0$, and $(r+k)^3 > 0$, so $f'(r) > 0$. For $k < r$ we have $c^2 > 0$, $k - r < 0$, and $(r+k)^3 > 0$, so $f'(r) < 0$. Thus $f'(r)$ changes sign from positive to negative at $r = k$, making k a local maximum of $f(r)$ by the First Derivative Test.

Note: $f'(r)$ changes sign from negative to positive at $r = -k$, but $-k$ is not a local minimum of $f(r)$ since $f(r)$ is undefined at $r = -k$. In fact, $f(r)$ has a vertical asymptote at $r = -k$, with $\lim_{r \to -k} f(r) = -\infty$.

Answer: Local min(s) at $r =$ **NONE** and Local max(es) at $r =$ k

b. [4 points] Find all **global extrema** of $f(r)$ **on the interval** $[0, \infty)$. If there are none of a particular type, write NONE. Your answer may involve c and/or k.

Solution: The function $f(r)$ is continuous on $[0, \infty)$, so we just need to check the values of $f(r)$ at all its critical points in $[0, \infty)$, as well as the behavior of $f(r)$ at the endpoints of $[0, \infty)$. In part **a**., we found $r = k$ to be the only critical point of $f(r)$. Since

$$f(0) = 0, \qquad f(k) = \frac{c^2 k}{(2k)^2} > 0, \qquad \text{and} \qquad \lim_{r \to \infty} f(r) = 0,$$

we find that $f(r)$ has a global min of 0 on $[0, \infty)$ at the point $r = 0$, and a global max of $\frac{c^2 k}{4k^2} = \frac{c^2}{4k}$ on $[0, \infty)$ at the point $r = k$.

Answer: Global min(s) at $r =$ 0 and Global max(es) at $r =$ k

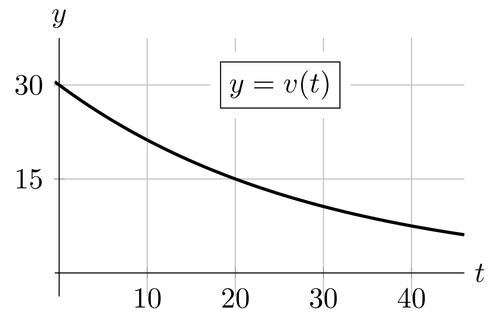
c. [3 points] Find values of the positive constants c and k so that the function $P = f(r)$ has an inflection point at $r = 4$ and a global maximum value of $P = 8$. *Show all work.*

Solution: Since $f''(4)$ exists, in order for $f(r)$ to have an inflection point at $r = 4$ we must have $f''(4) = 0$. This means $4 - 2k = 0$, so $k = 2$. In part **c**., we found $f(r)$ has a global maximum value of $\frac{c^2}{4k}$, so by setting $\frac{c^2}{4k}$ equal to 8 and plugging in $k = 2$, we find $c = 8$.

Answer: $c =$ 8 and $k =$ 2

11. [8 points]

A car runs out of gas and begins coasting on a highway. A graph of the car's velocity $v(t)$ in meters per second t seconds after it begins coasting is shown to the right.



Let $d(t)$ be the distance the car has traveled, in meters, since it started coasting, and let $L(t)$ be the linear approximation of $d(t)$ at $t = 20$.

a. [1 point] Approximately how far, in meters, did the car coast over the first 20 seconds after it began coasting? Give the best numerical estimate you can, based on the graph of $v(t)$.

Solution: We need to find the area of the region under the curve $v(t)$ between $t = 0$ and $t = 20$, which appears to be a bit less than 3 squares. Since each square has area 150, we get an answer of a bit less than 450 meters. *Any answer in the interval* $[400, 450]$ *received credit.*

Answer: approximately 433 meters

b. [2 points] What formula would you get for $L(t)$ if you used your estimate from part **a**?

Solution: We have $L(t) = d(20) + d'(20)(t - 20)$. Since $d'(t) = v(t)$ and $v(20) = 15$ from the graph, using our approximation $d(20) \approx 433$ from part **a.** we get

$$L(t) = 433 + 15(t - 20).$$

Answer: $L(t) \approx$ $433 + 15(t - 20)$

c. [3 points] In each row below, circle the **one** true statement.

Solution: Note that the graph of $d(t)$ is increasing and concave down.

(i) *circle one:* $L'(20) < d'(20)$ $L'(20) > d'(20)$ $\boxed{L'(20) = d'(20)}$

(ii) *circle one:* $\boxed{L'(15) < d'(15)}$ $L'(15) > d'(15)$ $L'(15) = d'(15)$

(iii) *circle one:* $L(15) < d(15)$ $\boxed{L(15) > d(15)}$ $L(15) = d(15)$

d. [2 points] Suppose that at $t = 0$, the car passes a truck on the highway. The truck is traveling in the same direction as the car, and is traveling at a constant speed of s meters per second. Eventually, the truck catches up with the coasting car and passes it at time $t = c$. How fast was the truck traveling?

Give an *exact answer* in terms of c and the function $v(t)$. Your answer may involve derivatives or integrals of $v(t)$, but it should **not** involve the function $d(t)$.

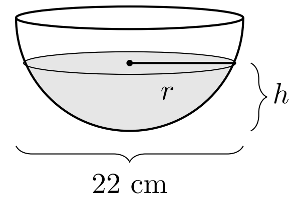
Solution: Between $t = 0$ and $t = c$, the truck travels a total of sc meters, while the car travels $\int_0^c v(t)\,dt$ meters. Therefore

$$sc = \int_0^c v(t)\,dt, \qquad \text{so} \qquad s = \frac{1}{c}\int_0^c v(t)\,dt.$$

Answer: $s = \frac{1}{c}\int_0^c v(t)\,dt$ meters per second

12. [9 points]

A hemispherical bowl of radius 11 cm is being filled with water at the constant rate of π cm^3/sec. When the depth of the water from the bottom of the bowl is h centimeters, the volume of the water in the bowl in cubic centimeters is given by



$$V = 11\pi h^2 - \frac{\pi}{3}h^3.$$

a. [4 points] At what rate is the depth of water in the bowl increasing when the depth of the water is 2 centimeters, in units of centimeters per second?

Solution: We are given that $\frac{dV}{dt} = \pi$, and we are asked to find $\frac{dh}{dt}$ when $h = 2$. Differentiating $V = 11\pi h^2 - \frac{\pi}{3}h^3$ with respect to t gives us

$$\frac{dV}{dt} = 22\pi h\frac{dh}{dt} - \pi h^2\frac{dh}{dt} = \pi h(22-h)\frac{dh}{dt}.$$

Plugging in $h = 2$ and using $\frac{dV}{dt} = \pi$, we get

$$\pi = \frac{dV}{dt} = \pi(2)(22-2)\frac{dh}{dt} = 40\pi\frac{dh}{dt}, \qquad \text{so} \qquad \frac{dh}{dt} = \frac{\pi}{40\pi} = \frac{1}{40}.$$

Answer: 1/40 cm/sec

b. [5 points] As the bowl fills up, the circular surface of the water grows larger. At what rate is the area of the water's surface increasing when the depth of the water is 2 centimeters (as in part **a**)? *Include units.*

Solution: Let r be the radius, and A the area, of the circular surface of the water. We must find $\frac{dA}{dt}$ when $h = 2$. Differentiating both sides of $A = \pi r^2$ with respect to t, we get

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}. \tag{1}$$

To find $\frac{dr}{dt}$, we need an equation relating r and h. From the picture, we see that the line segment labeled "r" and the segment of length $11-h$ from the center of the circular surface of the water to the center of the hemispherical bowl form two legs of a right triangle with hypotenuse 11 cm. Therefore by the Pythagorean Theorem we have

$$r^2 + (11-h)^2 = 11^2, \qquad \text{so} \qquad r^2 = 22h - h^2.$$

Differentiating this gives us

$$2r\frac{dr}{dt} = 22\frac{dh}{dt} - 2h\frac{dh}{dt}. \tag{2}$$

Now we can solve for $\frac{dA}{dt}$ when $h = 2$ by combining equations (1) and (2) and plugging in $h = 2$ and $\frac{dh}{dt} = \frac{1}{40}$ from part **a**:

$$\begin{aligned}\frac{dA}{dt} &= 2\pi r\frac{dr}{dt} = \pi\left(22\frac{dh}{dt} - 2h\frac{dh}{dt}\right)\\ &= \pi\left(22\frac{1}{40} - 2(2)\frac{1}{40}\right) = \frac{18\pi}{40} \qquad \text{when } h = 2.\end{aligned}$$

Answer: $\frac{18\pi}{40}$ **cm^2/sec**