6. (12 points) Consider the function $f(x) = \sin(x^2)$.

(a) Explain what the difference quotient $\dfrac{\sin(\sqrt{\pi}^2) - \sin(0)}{\sqrt{\pi}}$ represents.

The difference quotient represents the slope of the secant line joining the points $(0, 0)$ and $(\sqrt{\pi}^2, \sin(\sqrt{\pi}^2))$–or, it gives the average rate of change of the function $f(x) = \sin(x^2)$ between $x = 0$ and $x = \sqrt{\pi}$.

(b) Write the **limit definition** for $f'(\sqrt{\pi})$ without using the symbol f. No need to numerically evaluate the limit or approximate $f'(\sqrt{\pi})$.

The definition gives

$$f'(\sqrt{\pi}) = \lim_{h \to 0} \frac{\sin((\sqrt{\pi} + h)^2) - \sin(\sqrt{\pi}^2)}{h}.$$

(c) Suppose that g is a new function defined as follows:

$$g(x) = \begin{cases} 2f(x) & x < \sqrt{\pi/2} \\ kx + 4 & x \geq \sqrt{\pi/2} \end{cases} \qquad \text{for } f(x) \text{ as above}$$

For what value of k is the function g continuous?

Since both f and $kx + 4$ are continuous, we need to assure that the functions meet at $x = \sqrt{\pi/2}$. Thus, we need to solve for k if

$$2\sin(\sqrt{\pi/2}^2) = k\sqrt{\pi/2} + 4.$$

This gives

$$2 = k\sqrt{\pi/2} + 4 \qquad \text{(since } \sin(\sqrt{\pi/2}^2) = 1\text{)},$$

so

$$k = \frac{-2}{\sqrt{\pi/2}}.$$