5. [10 points] Electric cars need large amounts of energy to operate. Most types of batteries, including those found in electric cars, have reduced capacities when discharged at higher rates. For the lithium-ion batteries used in the newest electric cars, this relationship can be expressed by the equation $C = f(I) = \frac{K}{I^n}$ where C is the working capacity of the battery in amp hours (Ah) given a discharge rate of I (with $n > 1$) measured in amps (A). The constant $K > 0$ is the rated capacity of the battery.

a. [5 points] Write a formula for the derivative of C at $I = 3$ using the limit definition of the derivative. You do not need to evaluate or simplify this expression.

Solution: The limit definition of the derivative in this case is

$$\begin{aligned}\frac{dC}{dI}(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}. \\ &= \lim_{h\to 0} \frac{K/(3+h)^n - K/3^n}{h}.\end{aligned}$$

b. [3 points] Is C increasing or decreasing at $I = 3$? Justify your answer.

Solution: The capacity C is decreasing at $I = 3$. To see this, look at the expression for the slope of the secant line on C between $I = 3$ and $I = 3 + h$ for small h:

$$\frac{K/(3+h)^n - K/3^n}{h}.$$

The expression is negative if $h > 0$ because $K/(3+h)^n < K/3^n$.
The expression is also negative if $h < 0$ because $K/(3+h)^n > K/3^n$.
Since all of the secant lines near $I = 3$ have negative slope, the derivative, which is a limit of the slopes of secant lines, must also be negative.

c. [2 points] What is the concavity of the graph of C at $I = 3$? Justify your answer.

Solution: The concavity of the graph of C at $I = 3$ is concave up. As stated in the problem, the capacity C is of the form $C = \frac{K}{I^n}$. This function is a scalar multiple of the power function I^{-n} which is concave up for $I > 0$ as long as $n > 0$.

Another approach is to use the power rule to differentiate the function twice. Starting with $C = f(I) = KI^{-n}$, we have

$$f'(I) = -nKI^{-n-1}$$

and

$$f''(I) = (-n)(-n-1)KI^{-n-2} = n(n+1)KI^{-n-2}.$$

Now for $I > 0$, $f''(I) > 0$ meaning $C = f(I)$ is concave up.