3. [12 points] A zombie plague has broken out in Ann Arbor. As a nurse in the University of Michigan hospital, you saw the person with the first case of the plague, patient zero.

a. [2 points] In order to keep track of the growing zombie population in Ann Arbor, you collected the following data:

Days after patient zero	0	6	9	12
Number of Zombies	1	9	27	81

Would a linear function or an exponential function be the best model? Why?

Solution: In order for a function to be linear, it must have constant slope. Using the table above, we can compute the slopes between subsequent points: $(9-1)/6 = 4/3$, $(27-9)/(9-6) = 6$, and $(81-27)/(12-9) = 18$. Not only are these values not constant, they are increasing, so a linear function would be a very bad model.
If instead we take ratios of outputs with the same change in input we have $27/9 = 3 = 81/27$. Since this value is constant, an exponential function will be a good model for this data.

b. [4 points] Write a function $Z(t)$ of the appropriate type to model the growth of the zombie population with t measured in days after patient zero.

Solution: From part a), we know we are looking for an exponential function, which can take the form $Z(t) = Ab^t$ or $Z(t) = Ae^{kt}$. Since $Z(0) = 1$, the value of A for both types of exponential equation will be 1. Let's find $Z(t)$ of the form b^t. Since $Z(6) = 9$ and $Z(9) = 27$, we can take the ratio of these two equations:

$$\frac{27}{9} = \frac{b^9}{b^6} \Rightarrow 3 = b^{9-6} = b^3 \Rightarrow b = 3^{1/3} \approx 1.44225.$$

Therefore we can write the function exactly as $Z(t) = 3^{t/3}$, or approximate with $Z(t) = 1.44225^t$.
Similarly, we can solve for $Z(t)$ in the form e^{kt}:

$$\frac{27}{9} = \frac{e^{9k}}{e^{6k}} \Rightarrow 3 = e^{3k} \Rightarrow \ln(3) = 3k \Rightarrow k = \frac{\ln(3)}{3}.$$

Then we have $Z(t) = e^{\frac{\ln(3)t}{3}}$.

c. [3 points] The population of North America is approximately 530,000,000 people. Using your model, how long will it take until all but one person are infected?

Solution: To solve for the time it takes for all but one of 530,000,000 people to get infected, we need to set $Z(t) = 529999999$ and solve for t.

$$\begin{aligned} 3^{t/3} &= 529999999 \\ \frac{t}{3}\ln(3) &= \ln(529999999) \\ t &= \frac{3\ln(529999999)}{\ln(3)} \approx 54.856 \text{ days} \end{aligned}$$

d. [3 points] Using your table, approximate the instantaneous rate of change of the zombie population on the ninth day.

Solution: To approximate the instantaneous rate of change from the table, we need to compute the average rate of change either between the sixth day and the ninth day, the ninth day and the twelfth day, or the sixth day and the twelfth day.

$$\begin{aligned} \text{Average rate of change between 6th and 9th day} &= \frac{27-9}{9-6} = 6 \text{ zombies per day} \\ \text{Average rate of change between 9th and 12th day} &= \frac{81-27}{12-9} = 18 \text{ zombies per day} \\ \text{Average rate of change between 6th and 12th day} &= \frac{81-9}{12-6} = 12 \text{ zombies per day} \end{aligned}$$