2. [13 points] After Blizzard left Arizona, Gabe the mouse found a large globe (a sphere) to climb. The globe has a diameter of 40 inches and it is attached to a 12-inch-long pole. Gabe starts at the base of the pole at point $P$. He climbs up to the bottom of the globe at point $Q$. He then climbs the globe along a semicircle until he stops at the top of the globe at point $R$ (see the diagram below). Note that the diagram is not drawn to scale.

Complete path: $PQGR$

a. [8 points] Assume that Gabe walks through the path at a velocity of 3 inches per second. Let $G(t)$ be Gabe’s height above the ground (in inches) $t$ seconds after he started his climb at point $P$. Find a piecewise-defined formula for $G(t)$. Be sure to include the domain for each piece.

Solution: From point P to Q: It takes the ant 4 seconds to climb 12 inches at a velocity of 3 inches per second. During that time, the ant climbs at a constant rate of 3 inches per seconds starting at the floor, hence $G(t) = 3t$ for $0 \leq t \leq 4$.

From point Q to R: The distance $L$ along the semicircle traveled by the ant is $L = \frac{1}{2}(2\pi R)$, where $R$ is the radius of the circle. In this case $R = 20$ inches, then $L = 20\pi$. Hence it takes the ant $T = \frac{L}{3} = \frac{20\pi}{3}$ seconds to go from point Q to R at a velocity of 3 inches per second. Its height is given by a sinusoidal function with midline at $k = 12 + 20 = 32$, amplitude $A = \frac{1}{2}(40) = 20$, period $P = 2T = \frac{40\pi}{3}$ and a minimum at $(4,12)$. Hence $G(t) = 32 - 20 \cos(B(t - 4))$. The constant $B = \frac{2\pi}{P} = \frac{2\pi}{\frac{40\pi}{3}} = \frac{3}{20}$ for $4 \leq t \leq 4 + T$. Hence

$$G(t) = \begin{cases} 
3t & \text{for } 0 \leq t \leq 4, \\
32 - 20 \cos\left(\frac{3}{20}(t - 4)\right) & \text{for } 4 \leq t \leq 4 + \frac{20\pi}{3} 
\end{cases}$$
b. [5 points] After climbing the globe, Gabe jumps onto a small ferris wheel. Let $H(t)$ be his height, in inches, above the ground $t$ seconds after Gabe jumped, where

$$H(t) = 12 + 9 \cos\left(\frac{\pi}{75}(t - 120)\right).$$

Find the smallest positive value of $t$ at which Gabe’s height above the ground is 10.5 inches. Clearly show each step of your algebraic work. Give your answer in exact form.

**Solution:**

\[
12 + 9 \cos\left(\frac{\pi}{75}(t - 120)\right) = 10.5
\]

\[
\cos\left(\frac{\pi}{75}(t - 120)\right) = -\frac{1}{6}
\]

\[
\frac{\pi}{75}(t - 120) = \cos^{-1}\left(-\frac{1}{6}\right)
\]

\[
t_0 = 120 + \frac{75}{\pi} \cos^{-1}\left(-\frac{1}{6}\right)
\]

(smallest positive)

\[
t_{\text{ans}} = t_0 - P = \frac{75}{\pi} \cos^{-1}\left(-\frac{1}{6}\right) - 30.
\]

where the period of $H(t)$ is $P = \frac{2\pi}{75} = 150$. 

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Fall, 2017 Math 115 Exam 1 Problem 2 (globe) Solution