9. [9 points] Consider the rational function r defined by

$$r(x) = \frac{3\left(x-\sqrt{2}\right)(\pi x+7)^2(x+1)}{(x+1)\left(x-\sqrt{3}\right)}.$$

For all of the following parts of this problem, leave your answers in *exact* form.

a. [2 points] What is the domain of $r(x)$?

Solution: $x \neq -1, \sqrt{3}$.

b. [2 points] Find the *equations* of all vertical asymptotes of $r(x)$. If there are none, write NONE.

Solution: $x = \sqrt{3}$

c. [2 points] Let $p(x) = 3x^2 + 1.2x - 5$. Find the *equations* of all horizontal asymptotes of $\frac{r(x)}{p(x)}$. If there are none, write NONE. Show your work or reasoning to justify your answer.

Solution: Since $\frac{r(x)}{p(x)} = \frac{3\left(x-\sqrt{2}\right)(\pi x+7)^2(x+1)}{(x+1)\left(x-\sqrt{3}\right)(3x^2+1.2x-5)}$. Then its horizontal asymptote(s) can be found by finding

$$A = \lim_{x\to\infty} \frac{3\left(x-\sqrt{2}\right)(\pi x+7)^2(x+1)}{(x+1)\left(x-\sqrt{3}\right)(3x^2+1.2x-5)} \quad \text{and} \quad B = \lim_{x\to-\infty} \frac{3\left(x-\sqrt{2}\right)(\pi x+7)^2(x+1)}{(x+1)\left(x-\sqrt{3}\right)(3x^2+1.2x-5)}.$$

In order to find A and B, we need to notice that the leading terms of $3(x-\sqrt{2})(\pi x+7)^2(x+1)$ and $(x+1)\left(x-\sqrt{3}\right)(3x^2+1.2x-5)$ are $3(x)(\pi x)^2(x) = 3\pi^2x^4$ and $(x)(x)(3x^2) = 3x^4$ respectively. Hence $A = B = \pi^2$. Then the horizontal asymptote of $\frac{r(x)}{p(x)}$ is $y = \pi^2$.

d. [3 points] If $q(x) = \frac{2e^{kx}}{1+2^x}$, find all values of k so that $\lim_{x\to\infty} q(x) = 0$. If there are none, write NONE. Show your work or reasoning to justify your answer.

Solution: In order for $\lim_{x\to\infty} q(x) = 0$, the function $y = 2^x$ must dominate $y = e^{kx}$. This is true if the growth factor of $y = 2^x$ is larger than the one of $y = e^{kx}$. Hence we are looking for the values of k such that $2 > e^k$. Hence $k < \ln(2)$.