8. [10 points] Let A and B be **positive** constants. The rational functions $y = P(x)$ and $y = Q(x)$ are given by the following formulas:

$$P(x) = \frac{5x(x-2)(Ax+1)^2}{(3x^2+B)(x^2-9)} \qquad Q(x) = \frac{P(x)(x-3)}{x-2}$$

Your answers below may depend on the constants A and B and should be in exact form. You do not need to show your work.

a. [3 points] Find the zeros of the function $y = P(x)$. If P has no zeros write "NONE".

Solution: Setting $5x(x-2)(Ax+1)^2 = 0$ you get $x = 0$, $x - 2 = 0$ and $(Ax+1)^2 = 0$. This yields $x = 0$, 2 and $-\frac{1}{A}$.

Answer: $x = 0$, 2 and $-\frac{1}{A}$.

b. [2 points] What is the domain of $P(x)$?

Solution: The only points not in the domain of $P(x)$ are the solutions to $(3x^2+B)(x^2-9) = 0$. Solving $x^2 - 9 = 0$ we get $x = \pm 3$. If we set $3x^2 + B = 0$, we get $x^2 = -\frac{B}{3} < 0$. This is not possible for any value of x. Then the only solutions are $x = \pm 3$.

Answer: $x \neq -3, 3$.

c. [2 points] Find the *equation(s)* of the horizontal asymptote(s) of $y = P(x)$. If it has no horizontal asymptotes, write "NONE".

Solution: To find the end behavior of $P(x)$ we need to find the leading coefficient of the numerator and the denominator. The leading term of the numerator is $5x(x-2)(Ax+1)^2$ is $5x(x)(Ax)^2 = 5A^2x^4$. The leading term of $(3x^2+B)(x^2-9)$ is $(3x^2)(x^2) = 3x^4$. Hence $\lim_{x\to\infty} P(x) = \lim_{x\to\infty} \frac{5A^2x^4}{3x^4} = \frac{5A^2}{3}$. This limit is the same as $\lim_{x\to-\infty} P(x)$.

Answer: $y = \frac{5A^2}{3}$

d. [3 points] If $A = 1$, find the values of c where $\lim_{x\to c} Q(x)$ does not exist. If no such values of c exist, write "NONE".

Solution: $\lim_{x\to c} Q(x)$ exists for all c in the domain of $Q(x) = \frac{5x(x-2)(Ax+1)^2(x-3)}{(3x^2+B)(x^2-9)(x-2)}$. Hence we need to check the limits at $c = -3$, 2 and 3. At $c = 2$, $\lim_{x\to 2} Q(x) = \frac{-2(2A+1)^2}{B+12}$ and at $c = 3$, $\lim_{x\to 3} Q(x) = \frac{15(3A+1)^2}{6(B+27)}$ (both of these points are holes in the graph of $Q(x)$). At $c = -3$, $Q(x)$ has a vertical asymptote hence $\lim_{x\to -3} Q(x)$ does not exist.

Answer: $c = -3$