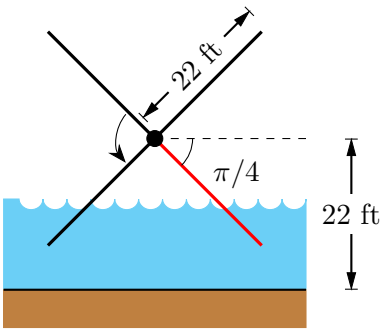
3. [11 points] You are standing by a river, watching three water wheels, each of which is rotating counterclockwise at a different but constant speed.

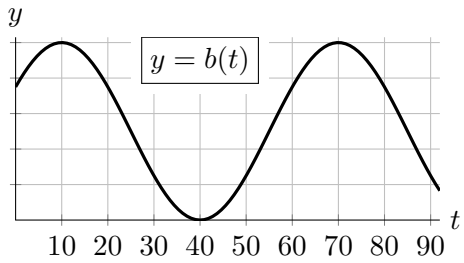
a. [4 points] The first water wheel takes 48 seconds to complete a full revolution. Each blade of the wheel is 22 feet long, and one of the blades is painted red. When each blade is at its lowest point, it just barely scrapes the bottom of the river. At the moment you begin watching, the red blade is exactly $\frac{\pi}{4}$ radians below the horizontal, as depicted to the right.



Write a formula for $r(t)$, the height, in feet, of the tip of the red blade above the bottom of the river t seconds after you begin watching.

Solution: The function $r(t)$ is sinusoidal.
Period: 48 seconds, Max: 44 ft, Min: 0 ft, Amplitude: 22 (the blades are 22 feet long)
Midline: $y = 22$ (the center of the wheel is 22 feet above the bottom of the river)
If the red blade were initially positioned at the horizontal instead of $\pi/4$ radians below it then the formula for $r(t)$ would be $22 + 22\sin\left(\frac{2\pi}{48}t\right)$.
However, the red blade is actually at the horizontal after 6 seconds (one eighth of a period), so one formula is $r(t) = 22 + 22\sin\left(\frac{2\pi}{48}(t-6)\right) = 22 + 22\sin\left(\frac{\pi}{24}t - \frac{\pi}{4}\right)$.
Other equivalent answers include: $r(t) = 22 + 22\cos\left(\frac{2\pi}{48}(t-18)\right) = 22 + 22\cos\left(\frac{\pi}{24}t - \frac{3\pi}{4}\right)$ and $r(t) = 22 - 22\cos\left(\frac{2\pi}{48}(t+6)\right) = 22 - 22\cos\left(\frac{\pi}{24}t + \frac{\pi}{4}\right)$.

b. [4 points] Now you begin watching the second water wheel, which has one blade painted blue. Let $b(t)$ be the height, in feet, of the tip of the blue blade above the bottom of the river t seconds after you begin watching. A portion of the graph of $b(t)$ is shown. Note that the scale on the y-axis is unknown.



The first time the blue blade reaches the water, since you began watching, is at $t = 28$.

i. At what time t does the tip of the blue blade leave the water?

ii. At what time t does the tip of the blue blade enter the water a second time?

Solution: i. Let w be the time when the blade leaves the water, a number larger than 40. Since the graph is symmetric about the line $t = 40$, the distance from w to 40 is the same as the distance from 40 to 28, the time when the blade enters the water. In other words, the blade reaches the bottom of the wheel halfway between when it enters and exits the water.

$$w - 40 = 40 - 28 \quad \text{so} \quad w = 52$$

Alternatively, since the graph peaks at $t = 10$ and $t = 70$, the distance from 70 to w is the same as the distance from 28 to 10: $70 - w = 28 - 10$, so again, $w = 52$.
ii. Since the first peak of the function is at $t = 10$ and the second peak of the function is at $t = 70$, the period of the function is $70 - 10 = 60$. The tip of the blue blade enters the water again after one full period has elapsed, at time $t = 28 + 60 = 88$.

c. [3 points] Finally, the third water wheel has a blade painted yellow, and you have determined that the height, in feet, of the tip of this blade above the bottom of the river t seconds after you began watching is given by

$$40 + 35\sin(Bt),$$

where B is some nonzero constant.

i. What is the length, in feet, of this yellow blade?

ii. How many feet above the bottom of the river is the center of this water wheel?

Solution:
i. The length of the yellow blade is the amplitude of this sinusoidal function, 35.
ii. The distance from the bottom of the river to the center of the wheel is the midline of the sinusoidal function, 40.