8. [11 points] Consider the rational function $g(x) = \dfrac{(x-12)(x-7)(x-2)}{(2x-4)(x-3)(x-5)}$.

a. [2 points] What are the vertical asymptotes of the function $g(x)$?

Solution: The vertical asymptotes of $g(x)$ are $x = 3$ and $x = 5$.
(Note that the numerator of $g(x)$ is equal to 0 when $x = 12$, $x = 7$, and $x = 2$, while the denominator is equal to 0 when $x = 2$, $x = 3$, and $x = 5$. Now $g(x) = \frac{(x-12)(x-7)}{2(x-3)(x-5)}$ when $x \neq 2$ so $\lim_{x\to 2} g(x)$ exists. Therefore $g(x)$ has a "hole" at $x = 2$ rather than a vertical asymptote.)

b. [2 points] What are the vertical asymptotes of the function $\dfrac{1}{g(x)}$?

Solution: The vertical asymptotes of $\dfrac{1}{g(x)}$ are $x = 7$ and $x = 12$.
(The function $\frac{1}{g(x)}$ is obtained from the function $g(x)$ by swapping its numerator and denominator. Therefore, the vertical asymptotes of $\frac{1}{g(x)}$ are the zeroes of $g(x)$, namely, $x = 7$ and $x = 12$.)

The piecewise function $h(x)$ is defined as follows, where $g(x)$ is as above, where $f(x)$ is from Problem 7 above, and where B is a nonzero constant.

$$h(x) = \begin{cases} \dfrac{e^{2x}}{x^2} & x \leq 3 \\ B \cdot f(x) & 3 < x \leq 6 \\ g(x) & 6 < x \end{cases}$$

c. [3 points] Find an *exact* value of B for which the function $h(x)$ is continuous at $x = 3$. Show your work.

Solution: Note that $h(3) = \lim_{x\to 3^-} h(x) = \dfrac{e^6}{9}$. while $\lim_{x\to 3^+} B \cdot f(x) = B \cdot f(3) = -B$. In order for $f(x)$ to be continuous at $x = 3$, these limits must be equal to each other, so $B = -\dfrac{e^6}{9}$.

Evaluate each of the expressions below. If a limit diverges to ∞ or $-\infty$ or if the limit does not exist for any other reason, write DNE.

d. [2 points] $\lim_{x\to\infty} h(x)$

Solution: Since $h(x) = g(x)$ when $x > 6$, this limit is the same as $\lim_{x\to\infty} g(x)$. The limit of this rational function is determined by the leading terms in the numerator and denominator. The leading term of the numerator is x^3, and the leading term of the denominator is $2x^3$. So

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x^3}{2x^3} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}.$$

e. [2 points] $\lim_{x\to -\infty} h(x)$

Solution:

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{e^{2x}}{x^2} = 0.$$

(The first equality holds since $h(x) = \frac{e^{2x}}{x^2}$ when $x \leq 3$. As x grows without bound in the negative direction, e^{2x} approaches 0 while x^2 grows without bound. So $\lim_{x\to-\infty} \frac{e^{2x}}{x^2} = 0$.)