8. [11 points] Consider the rational function \( g(x) = \frac{(x - 12)(x - 7)(x - 2)}{(2x - 4)(x - 3)(x - 5)} \).

a. [2 points] What are the vertical asymptotes of the function \( g(x) \)?

Solution: The vertical asymptotes of \( g(x) \) are \( x = 3 \) and \( x = 5 \).
(Note that the numerator of \( g(x) \) is equal to 0 when \( x = 12, x = 7, \) and \( x = 2 \), while the denominator is equal to 0 when \( x = 2, x = 3, \) and \( x = 5 \). Now \( g(x) = \frac{(x-12)(x-7)}{2(x-3)(x-5)} \) when \( x \neq 2 \) so \( \lim_{x \to 2} g(x) \) exists. Therefore \( g(x) \) has a “hole” at \( x = 2 \) rather than a vertical asymptote.)

b. [2 points] What are the vertical asymptotes of the function \( \frac{1}{g(x)} \)?

Solution: The vertical asymptotes of \( \frac{1}{g(x)} \) are \( x = 7 \) and \( x = 12 \).
(The function \( \frac{1}{g(x)} \) is obtained from the function \( g(x) \) by swapping its numerator and denominator. Therefore, the vertical asymptotes of \( \frac{1}{g(x)} \) are the zeroes of \( g(x) \), namely, \( x = 7 \) and \( x = 12 \).)

The piecewise function \( h(x) \) is defined as follows, where \( g(x) \) is as above, where \( f(x) \) is from Problem 7 above, and where \( B \) is a nonzero constant.

\[
h(x) = \begin{cases} 
\frac{e^{2x}}{x^2} & x \leq 3 \\
B \cdot f(x) & 3 < x \leq 6 \\
g(x) & 6 < x 
\end{cases}
\]

c. [3 points] Find an exact value of \( B \) for which the function \( h(x) \) is continuous at \( x = 3 \). Show your work.

Solution: Note that \( h(3) = \lim_{x \to 3^-} h(x) = \frac{e^6}{9} \), while \( \lim_{x \to 3^+} B \cdot f(x) = B \cdot f(3) = -B \). In order for \( f(x) \) to be continuous at \( x = 3 \), these limits must be equal to each other, so \( B = -\frac{e^6}{9} \).

Evaluate each of the expressions below. If a limit diverges to \( \infty \) or \( -\infty \) or if the limit does not exist for any other reason, write DNE.

d. [2 points] \( \lim_{x \to \infty} h(x) \)

Solution: Since \( h(x) = g(x) \) when \( x > 6 \), this limit is the same as \( \lim_{x \to \infty} g(x) \). The limit of this rational function is determined by the leading terms in the numerator and denominator. The leading term of the numerator is \( x^3 \), and the leading term of the denominator is \( 2x^3 \). So

\[
\lim_{x \to \infty} h(x) = \lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{x^3}{2x^3} = \lim_{x \to \infty} \frac{1}{2} = \frac{1}{2}.
\]

e. [2 points] \( \lim_{x \to -\infty} h(x) \)

Solution:

\[
\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} \frac{e^{2x}}{x^2} = 0.
\]
(The first equality holds since \( h(x) = \frac{e^{2x}}{x^2} \) when \( x \leq 3 \). As \( x \) grows without bound in the negative direction, \( e^{2x} \) approaches 0 while \( x^2 \) grows without bound. So \( \lim_{x \to -\infty} \frac{e^{2x}}{x^2} = 0 \).)

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