7. [15 points] In each of the following problems, give a *formula* for a function whose domain is all real numbers, with *all* of the indicated properties. If there is no such function, then write "NO SUCH FUNCTION EXISTS". You do not need to show your work.

a. [6 points] A sinusoidal function $P(t)$ with the following three properties:
(i.) The period of the graph of $P(t)$ is 7.
(ii.) The graph of $P(t)$ attains a maximum value at the point $(1, 20)$.
(iii.) The graph of $P(t)$ attains a minimum value at the point $(-2.5, -6)$.

Solution: If we move the function to the left by 1, we get a cosine function with period 7, amplitude 13, and vertical shift 7. Call this shifted function $\widetilde{P}(t)$. Since

$$\widetilde{P}(t) = 13\cos\left(\frac{2\pi}{7}t\right) + 7$$

and $P(t)$ is $\widetilde{P}(t)$ shifted right by one, we get

$$P(t) = 13\cos\left(\frac{2\pi}{7}(t-1)\right) + 7$$

b. [3 points] A function $h(x)$ with the following two properties:
(i.) $h(x)$ is concave down for all x
(ii.) $h(x) > 0$ for all x.

Solution: No such function exists. If the function is decreasing at some point, it will decrease faster and faster until it touches the x-axis. If the function is increasing at some point, similar logic applies (reading right to left, rather than left to right). The only other possibility is that the function is not increasing or decreasing anywhere, but then it would just be a horizontal line with no concavity.

c. [3 points] A function $j(x)$ with the following two properties:
(i.) $j(x)$ is decreasing for all x.
(ii.) $j(x)$ is concave up for all x.

Solution: One example of such a function that we have encountered is an exponential decay function, for instance $j(x) = e^{-x}$.

d. [3 points] A *rational* function $\ell(x)$ with the following two properties:
(i.) $\ell(0) = 2$.
(ii.) The line $y = 2$ is a horizontal asymptote to the graph of $\ell(x)$.

Solution: We want the function to be defined everywhere, so we need the denominator not to have any roots. For instance, we can take the polynomial $x^2 + 1$ for the denominator. If we make this choice, the numerator will need to have degree 2 (so that the horizontal asymptote exists and is not the line $y = 0$) and first coefficient $2x^2$. Now we rig the numerator in whatever way we like to get $\ell(0) = 2$. One answer is

$$\ell(x) = \frac{2x^2 + 2x + 2}{x^2 + 1}$$