9. [7 points] Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} 
xe^{Ax} + B & \text{if } x < 3 \\
C(x-3)^2 & \text{if } 3 \leq x \leq 5 \\
\frac{130}{x} & \text{if } x > 5.
\end{cases}$$

Suppose $f(x)$ satisfies all of the following:

- $f(x)$ is continuous at $x = 3$.
- $\lim_{{x \to 5^+}} f(x) = 2 + \lim_{{x \to 5^-}} f(x)$.
- $\lim_{{x \to -\infty}} f(x) = -4$.

Find the values of $A$, $B$, and $C$.

Show your work. You must give exact answers. Do not use decimal approximations.

For example, $0.333333333$ would not be an acceptable answer if the answer were $\frac{1}{3}$.

Solution: Because $f(x)$ is continuous at $x = 3$ (the first property),

$$\lim_{{x \to 3^-}} f(x) = \lim_{{x \to 3^+}} f(x).$$

So we have $3e^{3A} + B = C(3-3)^2 = 0$ and thus $3e^{3A} = -B$ (*).

Now, by the second property, we have

$$\lim_{{x \to 5^+}} f(x) = 2 + \lim_{{x \to 5^-}} f(x),$$

so

$$\frac{130}{5} = 2 + C(5-3)^2$$

$$26 = 2 + 4C$$

$$24 = 4C$$

$$6 = C$$

Thus $C = 6$.

Note that if $\lim_{{x \to -\infty}} xe^{Ax}$ exists, then it is equal to 0 (and $A < 0$). By the third property, we therefore see that

$$-4 = \lim_{{x \to -\infty}} f(x) = \lim_{{x \to -\infty}} (xe^{Ax} + B) = 0 + B = B.$$

So, $B = -4$, and using equation (*) above, we see that $3e^{3A} = -(-4)$ so $e^{3A} = \frac{4}{3}$ and

$$A = \frac{1}{3} \ln\left(\frac{4}{3}\right)$$

Answer: $A = \frac{1}{3} \ln\left(\frac{4}{3}\right)$, $B = -4$, and $C = 6$. 