9. [7 points] Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} xe^{Ax} + B & \text{if } x < 3 \\ C(x-3)^2 & \text{if } 3 \le x \le 5 \\ \dfrac{130}{x} & \text{if } x > 5. \end{cases}$$

Suppose $f(x)$ satisfies all of the following:

- $f(x)$ is continuous at $x = 3$.
- $\lim_{x\to 5^+} f(x) = 2 + \lim_{x\to 5^-} f(x)$.
- $\lim_{x\to -\infty} f(x) = -4$.

Find the values of A, B, and C.
Show your work. You must give exact answers. Do not use decimal approximations.
For example, 0.333333333 *would* <u>*not*</u> *be an acceptable answer if the answer were* $\frac{1}{3}$.

Solution: Because $f(x)$ is continuous at $x = 3$ (the first property), $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$. So we have $3e^{3A} + B = C(3-3)^2 = 0$ and thus $3e^{3A} = -B$ (*).

Now, by the second property, we have $\lim_{x\to 5^+} f(x) = 2 + \lim_{x\to 5^-} f(x)$, so

$$\begin{aligned} \frac{130}{5} &= 2 + C(5-3)^2 \\ 26 &= 2 + 4C \\ 24 &= 4C \\ 6 &= C \end{aligned}$$

Thus $\boxed{C = 6}$.

Note that if $\lim_{x\to -\infty} xe^{Ax}$ exists, then it is equal to 0 (and $A < 0$). By the third property, we therefore see that

$$-4 = \lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} (xe^{Ax} + B) = 0 + B = B.$$

So, $\boxed{B = -4}$, and using equation (*) above, we see that $3e^{3A} = -(-4)$ so $e^{3A} = \frac{4}{3}$ and $\boxed{A = \frac{1}{3}\ln(4/3)}$.

Answer: $A =$ $\frac{1}{3}\ln\left(\frac{4}{3}\right)$, $B =$ -4, and $C =$ 6