5. [9 points]

a. [4 points] Each year, a lake reaches its maximum temperature of 76 degrees Fahrenheit (°F) on September 1, and its minimum temperature of 40°F on March 1. Write a formula for a sinusoidal function $T(m)$ modeling the lake's temperature, in °F, m months after January 1.

Solution: We know that the amplitude must be half of $76 - 40 = 36$, or 18, and that the vertical shift must be $40 + 18 = 58$. We also see that the period is 12 months, and that a minimum occurs on March 1, which is two months after January 1.

Answer: $T(m) = -18\cos\left(\frac{\pi}{6}(m-2)\right) + 58$

b. [5 points] The depth $D(m)$ of this lake, in feet, can also be represented by a sinusoidal function, namely

$$D(m) = 985 + 20\sin\left(\frac{\pi}{6}m\right),$$

where m is the time in months after January 1. Find the amount of time, in months, each year when the depth of the lake is at least 1000 feet. Give your answer in **exact form**.

Solution: We set the equation equal to 1000 and solve:

$$\begin{aligned}
985 + 20\sin\left(\frac{\pi}{6}m\right) &= 1000 \\
\sin\left(\frac{\pi}{6}m\right) &= \frac{3}{4}, \quad \text{so that one possible solution is} \\
\frac{\pi}{6}m &= \arcsin\left(\frac{3}{4}\right) \\
m &= \frac{6}{\pi}\arcsin\left(\frac{3}{4}\right) \quad \text{which is } \approx 1.6197.
\end{aligned}$$

The next solution, which could also be found using symmetries of the graph of $D(m)$, is

$$\begin{aligned}
\frac{\pi}{6}m &= \pi - \arcsin\left(\frac{3}{4}\right) \\
m &= \frac{6}{\pi}\left(\pi - \arcsin\left(\frac{3}{4}\right)\right) \quad \text{which is } \approx 4.3803.
\end{aligned}$$

Thus our final answer is $\frac{6}{\pi}\left(\pi - \arcsin\left(\frac{3}{4}\right)\right) - \frac{6}{\pi}\arcsin\left(\frac{3}{4}\right) = 6 - \frac{12}{\pi}\arcsin\left(\frac{3}{4}\right)$.

Answer: $6 - \frac{12}{\pi}\arcsin\left(\frac{3}{4}\right)$