3. [11 points] A pilot is flying in an air show. Let $A(t)$ be her altitude, in feet (ft) above the ground, $t$ seconds (sec) after takeoff. Some values of $A(t)$ are shown in the table below, and there is one missing value, denoted by “?”.

<table>
<thead>
<tr>
<th>$t$</th>
<th>5</th>
<th>22</th>
<th>23</th>
<th>60</th>
<th>60.1</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(t)$</td>
<td>300</td>
<td>1100</td>
<td>1400</td>
<td>400</td>
<td>?</td>
<td>1200</td>
</tr>
</tbody>
</table>

a. [3 points] Use the table to give the best possible estimate of $A'(22)$. Make sure to include the relevant units as part of your answer.

**Solution:** The best possible estimate of $A'(22)$ is obtained when we calculate the average rate of change over the smallest available interval containing $t = 22$. In this case, the smallest available interval is $[22, 23]$, and so we compute:

$$A'(22) \approx \text{Average rate of change over } [22, 23]$$

$$= \frac{1400 - 1100}{23 - 22}$$

$$= \frac{300}{1}$$

$$= 300 \text{ feet per second.}$$

b. [3 points] Suppose that $A'(60) = 550$. Give an approximate value for the missing entry in the table. Make sure to include the relevant units as part of your answer.

**Solution:** The equation $A'(60) = 550$ means that, when $\varepsilon$ is a small number, we have $A(60 + \varepsilon) \approx 400 + 550 \cdot \varepsilon$. The missing entry in the table is at $t = 60.1$, so here we may take $\varepsilon$ to be the number 0.1.

Then the equation $A'(60) = 550$ tells us that the missing entry $A(60.1)$ in the table is approximately $A(60) + 550 \cdot 0.1 = 400 + 55 = 455$ feet.

c. [5 points] The pilot flies in a different air show a week later. Let $B(t)$ be her altitude, in feet (ft) above the ground, $t$ seconds (sec) after takeoff. A graph of $B(t)$ is shown below.

(Reduced scale for solutions)
Let the quantities I-V be defined as follows:

I. The number 0.
II. The pilot’s average velocity, in ft/sec, between \( t = 15 \) and \( t = 50 \).
III. The pilot’s instantaneous velocity, in ft/sec, at \( t = 55 \).
IV. The pilot’s average velocity, in ft/sec, between \( t = 50 \) and \( t = 90 \).
V. The pilot’s instantaneous velocity, in ft/sec, at \( t = 85 \).

List the quantities I-V in increasing order.

**Solution:** Since \( B(15) < B(50) \) and \( B'(85) > 0 \), we see that II and V are greater than I. Since \( B(50) > B(90) \) and \( B'(55) < 0 \), we see that III and IV are less than I. Therefore our ordering is

\[(III \text{ or } IV) < (III \text{ or } IV) < I < (II \text{ or } V) < (II \text{ or } V)\].

Glancing at the graph, it appears that II is a shallow positive slope, while V is a steep positive slope. It also appears that IV is a shallow negative slope, while III is a steep negative slope. This suggests the answer

\[III < IV < I < II < V\].

For the purpose of these solutions, we will verify this answer more carefully, just to be sure.

We now decide whether II or V is greater. Observe that \( B(15) \) is about 750, and \( B(50) \) is a little less than 900. Therefore the pilot’s average velocity between \( t = 15 \) and \( t = 50 \) (option II) is no more than \( \frac{900-750}{50-15} = \frac{150}{35} < \frac{150}{30} = 5 \text{ ft/sec} \). Observe that \( B'(85) \) (option V) appears very large, almost certainly greater than 5. Indeed, we see that it must be larger than the pilot’s average velocity between \( t = 80 \) and \( t = 90 \). Since \( B(80) \) is less than 150 and \( B(90) \) is greater than 600, this average velocity is greater than \( \frac{600-150}{90-80} = \frac{450}{10} = 45 \text{ ft/sec} \). Since 45 > 5, we conclude that V is greater than II.

We now decide whether III or IV is greater. Observe that \( B(50) \) is a little less than 900 and \( B(90) \) is more than 600. Therefore the pilot’s average velocity between \( t = 50 \) and \( t = 90 \) (option IV) is greater than \( \frac{600-900}{90-50} = \frac{-300}{40} = -\frac{15}{2} > -8 \text{ ft/sec} \). Observe that \( B'(55) \) (option III) appears likely to be much less than -8. Indeed, we see that it must be less than the pilot’s average velocity between \( t = 50 \) and \( t = 60 \). Since \( B(50) \) is greater than 750 and \( B(60) \) is less than 150, this average velocity is less than \( \frac{150-750}{60-50} = \frac{-600}{10} = -60 \text{ ft/sec} \). Since -60 < -8, we conclude that III is less than IV.