3. [11 points] A pilot is flying in an air show. Let $A(t)$ be her altitude, in feet (ft) above the ground, t seconds (sec) after takeoff. Some values of $A(t)$ are shown in the table below, and there is one missing value, denoted by "?".

t	5	22	23	60	60.1	70
$A(t)$	300	1100	1400	400	?	1200

a. [3 points] Use the table to give the best possible estimate of $A'(22)$. Make sure to include the relevant units as part of your answer.

Solution: The best possible estimate of $A'(22)$ is obtained when we calculate the average rate of change over the smallest available interval containing $t = 22$. In this case, the smallest available interval is $[22, 23]$, and so we compute:

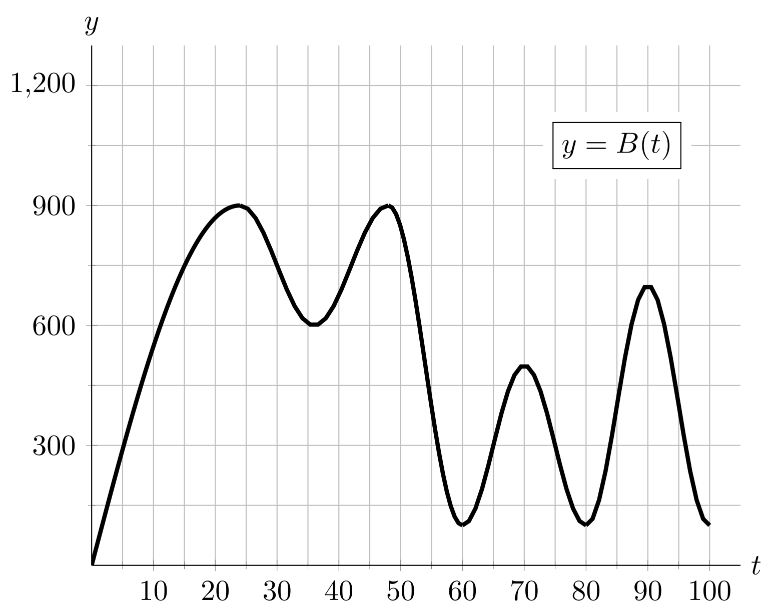
$$\begin{aligned} A'(22) &\approx \text{Average rate of change over } [22, 23] \\ &= \frac{1400 - 1100}{23 - 22} \\ &= \frac{300}{1} \\ &= 300 \text{ feet per second.} \end{aligned}$$

b. [3 points] Suppose that $A'(60) = 550$. Give an approximate value for the missing entry in the table. Make sure to include the relevant units as part of your answer.

Solution: The equation $A'(60) = 550$ means that, when ε is a *small* number, we have $A(60 + \varepsilon) \approx 400 + 550 \cdot \varepsilon$. The missing entry in the table is at $t = 60.1$, so here we may take ε to be the number 0.1.

Then the equation $A'(60) = 550$ tells us that the missing entry $A(60.1)$ in the table is approximately $A(60) + 550 \cdot 0.1 = 400 + 55 = 455$ feet.

c. [5 points] The pilot flies in a different air show a week later. Let $B(t)$ be her altitude, in feet (ft) above the ground, t seconds (sec) after takeoff. A graph of $B(t)$ is shown below.
(Reduced scale for solutions)



Let the quantities I-V be defined as follows:

I. The number 0.

II. The pilot's average velocity, in ft/sec, between $t = 15$ and $t = 50$.

III. The pilot's instantaneous velocity, in ft/sec, at $t = 55$.

IV. The pilot's average velocity, in ft/sec, between $t = 50$ and $t = 90$.

V. The pilot's instantaneous velocity, in ft/sec, at $t = 85$.

List the quantities I-V in increasing order.

Solution: Since $B(15) < B(50)$ and $B'(85) > 0$, we see that II and V are greater than I. Since $B(50) > B(90)$ and $B'(55) < 0$, we see that III and IV are less than I. Therefore our ordering is

$$\text{(III or IV)} < \text{(III or IV)} < \text{I} < \text{(II or V)} < \text{(II or V)}.$$

Glancing at the graph, it appears that II is a shallow positive slope, while V is a steep positive slope. It also appears that IV is a shallow negative slope, while III is a steep negative slope. This suggests the answer

$$\text{III} < \text{IV} < \text{I} < \text{II} < \text{V}.$$

For the purpose of these solutions, we will verify this answer more carefully, just to be sure.

We now decide whether II or V is greater. Observe that $B(15)$ is about 750, and $B(50)$ is a little less than 900. Therefore the pilot's average velocity between $t = 15$ and $t = 50$ (option II) is no more than $\frac{900-750}{50-15} = \frac{150}{35} < \frac{150}{30} = 5$ ft/sec. Observe that $B'(85)$ (option V) appears very large, almost certainly greater than 5. Indeed, we see that it must be larger than the pilot's average velocity between $t = 80$ and $t = 90$. Since $B(80)$ is less than 150 and $B(90)$ is greater than 600, this average velocity is greater than $\frac{600-150}{90-80} = \frac{450}{10} = 45$ ft/sec. Since $45 > 5$, we conclude that V is greater than II.

We now decide whether III or IV is greater. Observe that $B(50)$ is a little less than 900 and $B(90)$ is more than 600. Therefore the pilot's average velocity between $t = 50$ and $t = 90$ (option IV) is greater than $\frac{600-900}{90-50} = \frac{-300}{40} = \frac{-15}{2} > -8$ ft/sec. Observe that $B'(55)$ (option III) appears likely to be much less than -8. Indeed, we see that it must be less than the pilot's average velocity between $t = 50$ and $t = 60$. Since $B(50)$ is greater than 750 and $B(60)$ is less than 150, this average velocity is less than $\frac{150-750}{60-50} = \frac{-600}{10} = -60$ ft/sec. Since $-60 < -8$, we conclude that III is less than IV.