4. [12 points] Parts a. and b. below are unrelated.

a. [6 points] Suppose that the temperature in Staunton, Virginia, in degrees Fahrenheit (°F), can be modeled by a sinusoidal function \( S(t) \) where \( t \) is the time in months since January 1. Note that, for example, August 1 is seven months after January 1. A formula for \( S(t) \) is

\[
55 - 21 \cos \left( \frac{\pi}{6} t \right),
\]

i. Using this model, what is the coldest temperature in Staunton?

**Solution:** The midline of this sinusoidal function is at an output value of 55, and the amplitude is 21. Therefore the lowest value is \( 55 - 21 = 34 \) °F.

ii. Using this model, what is the average temperature over the entire year?

**Solution:** The midline of this sinusoidal function gives us the average temperature, and so the average temperature is 55°F.

iii. At what time \( t \) does the temperature first reach “room temperature” (68°F)? Give your final answer in exact form.

**Solution:** To find when the temperature first reaches room temperature, we must solve for \( t \) in the following equation:

\[
55 - 21 \cos \left( \frac{\pi}{6} t \right) = 68.
\]

Since we are looking for the first time the temperature reaches room temperature (and there is no horizontal shift from cosine in the given function), the arccosine function will give us our desired \( t \)-value. We therefore solve:

\[
55 - 21 \cos \left( \frac{\pi}{6} t \right) = 68
\]

\[
-21 \cos \left( \frac{\pi}{6} t \right) = 68 - 55
\]

\[
\cos \left( \frac{\pi}{6} t \right) = -13/21
\]

so one solution is given by \( \frac{\pi}{6} t = \arccos (-13/21) \)

\[
t = \frac{6}{\pi} \arccos (-13/21).
\]
b. [6 points] Suppose that a probe lands on some planet other than Earth, and that its recorded temperature, in degrees Fahrenheit, can be modeled by a sinusoidal function $P(a)$ where $a$ is the time in years since the probe landed. Note that the scale on the $y$-axis is unknown.

When the temperature is too cold, the probe is in a state of hibernation. The first time it enters hibernation is at $a = 27$.

i. At what time $a$ does the probe leave hibernation?

Solution: The probe will leave hibernation the next time after $a = 27$ that $P(a)$ is equal to $P(27)$. Since $P$ has a minimum at $a = 40$, its graph is symmetric about the vertical line $a = 40$, and so the probe will leave hibernation $40 - 27 = 13$ years after $a = 40$. That is to say, the probe will leave hibernation at $a = 53$.

ii. What is the period of $P(a)$?

Solution: We observe that the function $P$ has a maximum at $a = 10$ and another maximum at $a = 70$. Therefore the period of $P(a)$ is $70 - 10 = 60$.

iii. Use the period you found to calculate the next time at which the probe will enter hibernation.

Solution: The probe will next enter hibernation after one period of $P(a)$ has passed since the first time it entered hibernation. Therefore, the probe will next enter hibernation at $a = 27 + 60 = 87$. 