4. [5 points] Let $f(x)$, $g(x)$, and $h(x)$ be the functions defined for all real numbers by

$$f(x) = 2^{c+1}c^x, \qquad g(x) = e^c\cos(cx), \quad \text{and} \quad h(x) = \begin{cases} f(x) & x \le 0 \\ g(x) & x > 0 \end{cases}$$

where c is a nonzero constant. In each part below, find an exact value for the constant c so that the given condition holds. (*Your value for the constant c may be different in each part.*)

a. [1 point] The function $f(x)$ has a continuous decay rate of 15%.

Solution: If we write an exponential function in the form $P = P_0e^{-kx}$ with $k > 0$, then k is its continuous decay rate. So in order for $2^{c+1}c^x = 2^{c+1}e^{(\ln c)x}$ to have a continuous decay rate of 15%, we want $\ln c = -.15$, so $c = e^{-.15}$.

Answer: $e^{-0.15}$

b. [1 point] The function $g(x)$ has a period of 3.

Solution: The cosine function has a period of 2π, so $g(x)$ has a period of $\frac{2\pi}{c}$. So we solve $3 = \frac{2\pi}{c}$ to get $c = \frac{2\pi}{3}$.

Answer: $\frac{2\pi}{3}$

c. [3 points] The function $h(x)$ is continuous at zero.

Solution: Since $f(x)$ and $g(x)$ are both continuous, in order for $h(x)$ to be continuous at zero we need $f(0) = g(0)$. Since $f(0) = 2^{c+1}c^0 = 2^{c+1}$ and $g(0) = e^c\cos(0) = e^c$, this means we need $2^{c+1} = e^c$. Solving this for c gives

$$c \;=\; \ln e^c \;=\; \ln(2^{c+1}) \;=\; (c+1)\ln 2 \;=\; c\ln 2 + \ln 2.$$

Thus $c(1-\ln 2) = c - c\ln 2 = \ln 2$, so $c = \dfrac{\ln 2}{1-\ln 2}$.

Answer: $\frac{1-\ln 2}{\ln 2}$