7. [7 points] Consider the rational function $q(x) = \dfrac{5x(x+1)(x+3)^2}{(x-2)(x+1)^2(x+3)}$.

a. [2 points] Find all x-values at which the function $q(x)$ has a vertical asymptote.

Solution: After canceling common factors from the numerator and denominator to eliminate potential holes, we find that aside from a hole at $x = -3$, $q(x)$ has the same graph as

$$\frac{5x(x+3)}{(x-2)(x+1)},$$

which has vertical asymptotes at $x = 2$ and $x = -1$.

Answer: $q(x)$ has vertical asymptotes at $x =$ $2, -1$

b. [2 points] Find the following limits. If a limit diverges to ∞ or $-\infty$ or does not exist for any other reason, write DNE.

i. $\lim_{x\to\infty} q(x)$

Solution: Since the numerator and denominator of $q(x)$ have the same degree, $\lim_{x\to\infty} q(x)$ is the ratio of the leading coefficients of the numerator and denominator, which is $\frac{5}{1} = 5$.

Answer: 5

ii. $\lim_{x\to -3} q(x)$

Solution: $\lim_{x\to -3} q(x) = \lim_{x\to -3} \dfrac{5x(x+3)}{(x-2)(x+1)} = 0.$

Answer: 0

Suppose the piecewise function $g(x)$ is defined as follows, where $q(x)$ is as above, and k is a constant.

$$g(x) = \begin{cases} e - e^{kx^3} & x \le 1 \\ q(x) & x > 1 \end{cases}$$

c. [3 points] Find an *exact* value of k for which the function $g(x)$ is continuous at $x = 1$. Show your work.

Solution: The function $g(x)$ is continuous at $x = 1$ if $\lim_{x\to 1} g(x) = g(1)$. Since

$$g(1) = e - e^k = \lim_{x\to 1^-}(e - e^{kx^3}) = \lim_{x\to 1^-} g(x),$$

we just need to find k such that $e - e^k = \lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} q(x) = q(1) = \dfrac{5\cdot 2\cdot 16}{(-1)\cdot 4\cdot 4} = -10.$ Now we solve:

$$\begin{aligned} e - e^k &= -10 \\ e + 10 &= e^k \\ \ln(e+10) &= k. \end{aligned}$$

Answer: $k =$ $\ln(e+10)$