5. (10 points) The *trebuchet*, a medieval catapult driven by a falling, hinged counterweight, can be simulated with the use of mathematical models. The range of the projectile flung from the catapult at an angle θ is given by

$$R = \frac{2{v_0}^2 \sin\theta \cos\theta}{g},$$

where g is the constant acceleration due to gravity and v_0 is the constant representing the initial velocity of the projectile.

(a) Find the exact value of θ on the interval $0 \le \theta \le \pi/2$ that maximizes the range of the projectile.

$$R' = \frac{2v_0^2}{g}\left(\cos\theta\cos\theta + \sin\theta(-\sin\theta)\right)$$

$$= \frac{2v_0^2}{g}\left(\cos^2\theta - \sin^2\theta\right)$$

$R' = 0$ if $\cos^2\theta = \sin^2\theta$ or $\tan^2\theta = 1$

Thus, on $\left[0, \frac{\pi}{2}\right]$, $\theta = \frac{\pi}{4}$ is the only C.P.

NOTE: $R(0) = 0 = R\left(\frac{\pi}{2}\right)$. Therefore, $R\left(\frac{\pi}{4}\right)$ is the max.

& $R\left(\frac{\pi}{4}\right) = \frac{v_0^2}{g} > 0$

(b) What is the maximum range?

$$R\left(\frac{\pi}{4}\right) = \frac{2v_0^2}{g}\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{v_0^2}{g}$$