(4.) (12 points) Consider the function:

$$f(x) = e^{\frac{-(ax)^2}{2}}, \qquad \text{for } a \text{ a positive constant.}$$

The graph of $y = f(x)$ is the (in)famous "bell curve," which occurs frequently in statistics, and occasionally in heated political debates as well.

(a) Compute $f''(x)$. Show your work.

Use the chain rule:

$$f'(x) = e^{\frac{-(ax)^2}{2}} \cdot (-(ax)) \cdot a = (-a^2 x) \cdot e^{\frac{-(ax)^2}{2}}$$

Now use the product rule, together with the previous line:

$$f''(x) = -a^2 \cdot e^{\frac{-(ax)^2}{2}} + (-a^2 x) \cdot \left[(-a^2 x) \cdot e^{\frac{-(ax)^2}{2}} \right]$$

$$f''(x) = a^2 e^{\frac{-(ax)^2}{2}} (a^2 x^2 - 1)$$

(b) For which value of a does the function f have an inflection point at $x = 3$?

First let's find out where $f''(x) = 0$, since this is a prerequisite for an inflection point. Since $e^k > 0$ for any k and $a^2 \neq 0$, we need only find out where $a^2 x^2 - 1 = 0$. This happens when $x = \pm 1/a$. If $x = 3$ and a is positive, we must have $a = 1/3$. To assure that this is an inflection point of f, we can check the sign of $f''(x)$ to the left and right of $x = 3$. We see that $f''(x)$ is negative to the left of $x = 3$ and positive to the right of $x = 3$. Thus, the function changes from concave down to concave up at $x = 3$ when $a = 1/3$.