6.(9+4 points) (a) Find the values of a and b so that the function $f(x) = axe^{-bx}$ has a local maximum at the point (3, 12).

$$f'(x) = abe^{-bx} - abxe^{-bx}$$

This is 0 when $a - abx = 0$. Therefore, either $a = 0$ or $x = \frac{1}{b}$. If $a = 0$, then $f(x) = 0$ for all x so cannot have a maximum at $(3, 12)$. So it must be that $x = \frac{1}{b}$. Since we want this maximum to occur at $x = 3$, $b = \frac{1}{3}$. Now using that the y-value at the maximum is 12, we have $12 = 3ae^{-1}$. Therefore, $a = 4e$.
Now to see this is actually a maximum we must take the second derivative:

$$f''(x) = -\frac{8}{3}e^{-\frac{1}{3}x+1} + \frac{4}{9}xe^{-\frac{1}{3}x+1}.$$

At $x = 3$, $f''(3) = -\frac{4}{3} < 0$. So we see this is a maximum of f.

(b) Does f have any inflection points for $x > 0$? If so, for what value(s) of x? If not, how do you know? [Use the function you found for part (a) here. Show your work or your reasoning.]

We have from part (a) that

$$f''(x) = \left(-\frac{8}{3} + \frac{4}{9}x\right) e^{-\frac{1}{3}x+1}.$$

This is equal to 0 if and only if $x = 6$. For $x < 6$ we have $f''(x) < 0$ and for $x > 6$ we have $f''(x) > 0$. Therefore $x = 6$ is an inflection point.

7. (4+3+3 points) **(a)** Determine the tangent line approximation for $f(x) = \sin x$ near $x = \frac{\pi}{3}$ (i.e., 60°).

$$f\left(\tfrac{\pi}{3}\right) = \sin\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2}.$$
$$f'(x) = \cos x, \text{ so } f'\left(\tfrac{\pi}{3}\right) = \cos\left(\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}\right).$$
$$\text{Thus, near } x = \tfrac{\pi}{3},\ f(x) \approx \left(\tfrac{\sqrt{3}}{2}\right) + \tfrac{1}{2}\left(x - \tfrac{\pi}{3}\right).$$

(b) Use your answer from part (a) to give an approximation of $\sin\left(\frac{31}{90}\pi\right)$ without using your calculator. Note that $\frac{31}{90}\pi = 62°$. Show your work.

$$\sin\left(\frac{31}{90}\pi\right) \approx \left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{31\pi}{90} - \frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{\pi}{90}\right).$$

(c) Should your answer for part (b) be an over estimate or an under estimate? Justify your answer without indicating what $\sin\left(\frac{31}{90}\pi\right)$ is from your calculator.

The graph of $y = \sin x$ is concave down at $x = \frac{\pi}{3}$, so it is an overestimate as can easily been seen by sketching a graph of $\sin x$. [Or, could take the second derivative here and use the fact that $f''(\frac{\pi}{3}) < 0$.]