3. (12 points) For the function $f(x) = ax^4 - 3x^3$ with constant $a > 0$, use the techniques of calculus to answer the following. Show your work and proper justification for your answers.

(a) Determine all critical points of f. Classify each as a local maximum, a local minimum, or neither.

First, $f'(x) = 4ax^3 - 9x^2 = x^2(4ax - 9)$ which is everywhere defined. So the critical points are the zeroes which are $x = 0$ and $x = \frac{9}{4a}$. Since the long term behavior of f' is like $4ax^3$, we see that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > \frac{9}{4a}$.
For $0 < x < \frac{9}{4a}$, note that the sign of $f'(x)$ is determined by the sign of $4ax - 9$ because x^2 is always positive. Since $4ax - 9$ is just a line with positive slope (recall $a > 0$) and x-intercept $\frac{9}{4a}$ we must have that $4ax - 9 < 0$ for $x < \frac{9}{4a}$. It follows that $f'(x) < 0$ for $0 < x < \frac{9}{4a}$. Using the first derivative test, $x = \frac{9}{4a}$ is a local minimum and $x = 0$ is neither.

(b) Determine any global maxima or minima (if any).

The long run behavior of $f(x)$ is like ax^4 so we see that $f(x) \to \infty$ as $x \to \pm\infty$. Therefore $f(x)$ has no global maximum, but $x = \frac{9}{4a}$ is the global minimum.

(c) Determine all (if any) inflection points.

The second derivative, $f''(x) = 12ax^2 - 18x = x(12ax - 18)$. So the zeroes of the second derivative are $x = 0$ and $x = \frac{18}{12a} = \frac{3}{2a}$. Because the second derivative is a parabola opening upward, we conclude that $f''(x) < 0$ for $0 < x < \frac{3}{2a}$ and that $f''(x) > 0$ for $x < 0$ and $x > \frac{3}{2a}$. Thus both $x = 0$ and $x = \frac{18}{12a}$ are points of inflection.