4. (10 points) Every year pesticides used on adjacent agricultural land drain off into Lake Michigan. Eventually, scientists predict that the lake will become saturated with pesticides. As a result, the amount of pesticides in the lake $P(t)$ (in parts per million) is given as a function of time, t, in years since 2000, by

$$P(t) = a(1 - e^{-kt}) + b$$

where a, b and k are positive constants. Assume the saturation level of the lake for pesticides is 50 parts per million.

(a) If in the year 2000 the pesticide level of Lake Michigan was 5 parts per million, find a and b.

When k is a positive constant we have

$$\lim_{x\to\infty} e^{-kx} = \lim_{x\to\infty} \frac{1}{e^{kx}} = 0$$

and therefore,

$$\lim_{x\to\infty} P(x) = a(1-0) + b = a + b.$$

Since the saturation level is 50 parts per million, we get the equation $a + b = 50$. But also $P(0) = b = 5$ from the information given. It follows that $a = 45$.

(b) Find k if the pesticide level was increasing at a rate of 3 parts per million per year in the year 2000.

Since $P'(t) = 45ke^{-kt}$ we have that $P(0) = 45k$. So, $45k = 3$ hence, $k = \frac{1}{15}$.

(c) When the pesticide level reaches 30 parts per million, fish from the lake cannot be consumed by humans. In what year will the pesticide level in the lake reach 30 parts per million?

We must solve

$$P(t) = 45(1 - e^{-t/15}) + 5 = 30$$

for t. This equation becomes

$$e^{-t/15} = 1 - \frac{25}{45}$$

and therefore, $t \sim 12.16$ years. Thus, in the year 2012, the pesticide level should reach 30 ppm.