5. (10 points)

(a) If the graph of
$$\frac{a}{y} + x^2 + b \ln y = 6$$

goes through the point $(2, 1)$ for some implicitly defined y, find a.

Since $(2,1)$ is on the curve it must satisfy the defining equation. Therefore,

$$a + 4 + 0 = 6$$

so $a = 2$.

(b) Suppose $g(x) = -4x + 9$ is the equation of the tangent to the curve defined above at the point $(2, 1)$. Find b.

The tangent line has a slope of -4 which tells us that $\frac{dy}{dx} = -4$ at the point $(2, 1)$. We can compute $\frac{dy}{dx}$ explicitly:

$$-ay^{-2}\frac{dy}{dx} + 2x + by^{-1}\frac{dy}{dx} = 0.$$

Plugging in $a = 2$ and solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{-2x}{by^{-1} - 2y^{-2}}.$$

So when we plug in $x = 2$, $y = 1$ and $\frac{dy}{dx} = -4$ we get the equation:

$$-4 = \frac{-4}{b-2}$$

so that $b = 3$ follows.