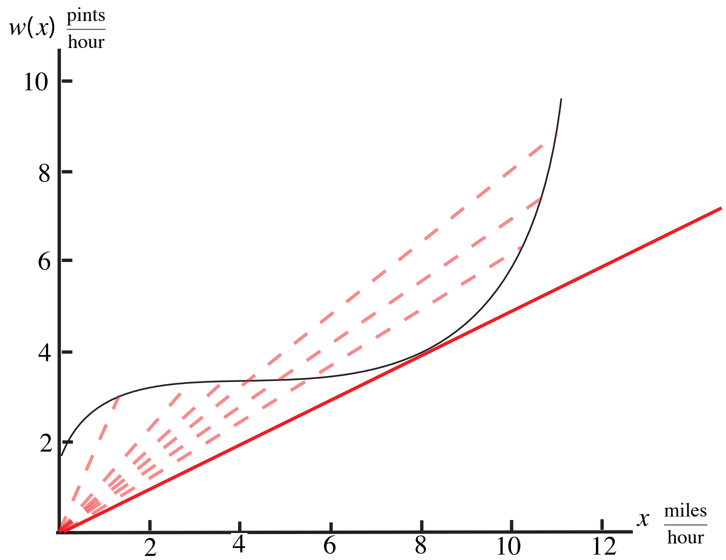
5. (12 points) Running a marathon takes a lot of energy. In order to keep up her energy level, Ellen drinks WolverineAid. Let $W(x)$ represent the number of pints of WolverineAid that Ellen must consume per hour when running at a rate of x miles per hour. The graph of $W(x)$ is given below.



(a) Let $C(x)$ represent Ellen's consumption of WolverineAid in pints per mile. How is $C(x)$ related to $W(x)$?

$$C(x) = \frac{W(x)}{x} \frac{\frac{\text{pints}}{\text{hour}}}{\frac{\text{miles}}{\text{hour}}} = \frac{W(x)}{x} \frac{\text{pints}}{\text{mile}}$$

(b) Use calculus to show that $C(x)$ has a critical point at $x = x_0$ when $W'(x_0) = C(x_0)$. (Show your work.)

Assuming $x \neq 0$ and using the quotient rule:

$$C'(x) = \frac{xW'(x) - W(x)}{x^2}$$

If $C'(x) = 0$ at $x = x_0$, then $x_0 W'(x_0) = W(x_0)$. Thus, $W'(x_0) = \dfrac{W(x_0)}{x_0} = C(x_0)$. Since $C'(x_0) = 0$ then x_0 is a critical point of $C(x)$.

(c) From the graph, approximate the pace that Ellen should run in order to get the most efficient use of the WolverineAid. Explain your answer.

To most efficiently use Wolverine Aid, Ellen should minimize $C(x)$. From part (a) we know that $C(x)$ can be represented as the slope of the line connecting the point $(x, W(x))$ and $(0, 0)$. The smallest slope occurs when $x \approx 8$. Also notice that this is indeed a critical point of $C(x)$ as the slope of the line from $(8, 4)$ is the same as the tangent line and consequently $W'(8) = \dfrac{W(8)}{8} = C(8)$.