7. (12 points) The flux F, in millilitres per second, measures how fast blood flows along a blood vessel. Poiseuille's Law states that the flux is proportional to the fourth power of the radius, R, of the blood vessel, measured in millimeters. In other words $F = kR^4$ for some positive constant k.

(a) Find a linear approximation for F as a function of R near $R = 0.5$. (Leave your answer in terms of k).

The linear approximation near $R = 0.5$ is given by $F(R) \approx F(0.5) + F'(0.5)(R - 0.5)$.

$$F(0.5) = k(0.5)^4 = \tfrac{k}{16},$$
$$F'(R) = 4kR^3$$

$$F'(0.5) = 4k(0.5)^3 = \tfrac{k}{2}$$

Thus for R near 0.5, $F(R)$ is given by:

$$\boxed{F(R) \approx \tfrac{k}{16} + \tfrac{k}{2}(R - 0.5)}$$

(b) A partially clogged artery can be expanded by an operation called an angioplasty, which widens the artery to increase the flow of blood. If the initial radius of the artery was 0.5mm, use your approximation from part (a) to approximate the flux when the radius is increased by 0.1mm.

To approximate when the radius increases by 0.1mm from 0.5mm we can evaluate the linear approximation from (a) at 0.6 giving:

$$\begin{aligned} F(0.6) &\approx \frac{k}{16} + \frac{k}{2}(0.6 - 0.5), \\ &= \frac{k}{16} + \frac{k}{20}, \\ &= \frac{9k}{80} = \boxed{(0.1125)\text{k millilitres per second}}. \end{aligned}$$

(c) Is the answer found in part (b) an under- or over-approximation? Justify your answer.

In order to determine if this is an over or an under approximation we use the second derivative:

$$F''(R) = 12kR^2$$

Since $k > 0$, $F''(R)$ is always positive, and the function is always concave up. Thus the linear approximation will always be an *under approximation*