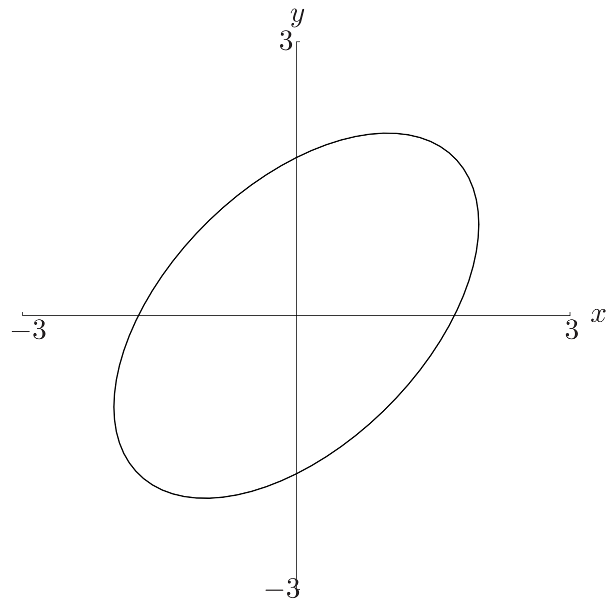
5. (12 points) The graph of

$$x^2 - xy + y^2 = 3$$

is a "tilted" ellipse (see the figure below). Among all points (x, y) on this graph, find the points that have the largest and smallest values of y. [Hint: Look at the figure to consider the conditions that would be true for y to take on largest or smallest values.] Be sure to show all work in order to justify your answer (*i.e.*. estimating points from a graph will not be sufficient).



Note that the largest and smallest values of y occur when $\frac{dy}{dx} = 0$. Taking the derivative of both sides with respect to x gives

$$2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0,$$

so

$$(2y - x)\frac{dy}{dx} = y - 2x,$$

which gives

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}.$$

Therefore $\frac{dy}{dx} = 0$ when $x = \frac{y}{2}$.

Substituting into the original equation:

$$\frac{y^2}{4} - \frac{y^2}{2} + y^2 = 3; \quad y^2 = 4; \quad \text{so } y = \pm 2.$$

Solving for x when $y = \pm 2$ gives the points $(1, 2)$ and $(-1, -2)$.

Largest y value is associated with the point: (1,2)

Smallest y value is associated with the point: (-1,-2)