2. In 1956, Marion Hubbert began a series of papers predicting that the United States' oil production would peak and then decline. Although he was criticized at the time, Hubbert's prediction was remarkably accurate. He modeled the annual oil production $P(t)$, in billions of barrels of oil, over time t, in years, as the *derivative* of the *logistic function* $Q(t)$ given below—*i.e.*, $Q'(t) = P(t)$. The function P is measured in years since the middle of 1910.

The function $Q(t)$ is given by

$$Q(t) = \frac{Q_0}{1 + ae^{-bt}}\text{ , where } a, b, Q_0 > 0. \tag{1}$$

For your convenience, the first and second derivatives of $Q(t)$ are given as well:

$$Q'(t) = -\frac{Q_0}{(1+ae^{-bt})^2}\left(-abe^{-bt}\right) = \frac{abQ_0e^{-bt}}{(1+ae^{-bt})^2},$$

and

$$Q''(t) = \frac{ab^2Q_0e^{-bt}}{(1+ae^{-bt})^3}\left[ae^{-bt} - 1\right].$$

(a) (2 points) Interpret, in the context of this problem, $P'(56)$.

$P'(56)$ is approximately the number of billions of barrels by which the United States' annual oil production increased from the middle of 1966 to the middle of 1967. (If $P'(56)$ is negative, then this represents a decrease in production during that time period.)

(b) (6 points) Determine the year of maximum annual production t_{max}. Your answer may involve all or some of the constants a, b, Q_0.

We have that $P'(t) = Q''(t) = \frac{ab^2Q_0e^{-bt}}{(1+ae^{-bt})^3}\left[ae^{-bt} - 1\right]$. The factor preceding the bracketed term is positive for all t. The bracketed term changes sign once, at $t = (1/b)\ln(a)$; so this is the only critical point of $P(t)$. The global maximimum occurs at this point, because $P'(t)$ is positive before that point and negative afterward. Thus, $t_{max} = \frac{1}{b}\ln a$.

(c) (2 points) Find the maximum annual production $P(t_{max})$. Again, your answer may involve all or some of the constants a, b, Q_0.

Using t_{max} from part (b), we get $P(t_{max}) = \frac{1}{4}bQ_0$.

(d) (2 points) In his 1962 paper, Hubbert studied the available data on oil production to date and concluded that $a = 46.8$, $b = 0.0687$, and $Q_0 = 170$ Bb (billion barrels). Using your results from part (b), when would Hubbert's curve predict the peak in US oil production? (The actual peak occurred in 1964.)

Using the results of part (b), we get $t_{max} = \frac{1}{0.0687}\ln(46.8) \approx 55.98$, which corresponds roughly to the middle of 1966.