4. [12 points] In preparation for the holidays, a local bookstore is planning to sell mugs of a variety of shapes. Suppose that the amount of liquid in a "UM" mug if filled to a depth of h cm is $L(h) = Uh(3M^2 - 3Mh + h^2)$ cm^3 for $U, M > 0$.

a. [4 points] Find and classify any critical points of L on the interval $(0, 5M)$.

Solution: Taking the derivative gives

$$L'(h) = U(3M^2 - 6Mh + 3h^2) = 3U(M^2 - 2Mh + h^2) = 3U(M - h)^2.$$

Thus, the only critical point occurs at $h = M$. Note that the factor $(M - h)^2$ is positive for all h, so the function is increasing to the left of $h = M$ and to the right of $h = M$. Thus, the critical point is neither a local maximum nor a local minimum.

b. [2 points] Determine any points of inflection of L on the interval $(0, 5M)$.

Solution: The second derivative, $L''(h) = -6U(M - h)$, shows a potential inflection point at $h = M$. The sign of the factor $-6U$ is always negative. The sign of the factor $(M - h)$ is positive to the left of $h = M$ and negative to the right. Thus, the product gives us $L''(h) < 0$ for $h < M$, and $L''(h) > 0$ for $h > M$, and the function changes from concave down to concave up at $h = M$, so L has an inflection point at $h = M$.

c. [6 points] Suppose you are pouring coffee into a "UM" mug at a rate of 15 cm^3 per second. At what rate is the depth of the coffee in the mug changing when the coffee reaches a depth of 4 cm in the mug?

Solution: Given $dL/dt = 15$ cm^3/s, we want to find dh/dt when $h = 4$ cm. We know

$$\frac{dL}{dt} = \frac{dL}{dh} \cdot \frac{dh}{dt},$$

so, when $h = 4$, we have

$$15 = 3U(M - 4)^2 \cdot \frac{dh}{dt}$$

and

$$\frac{dh}{dt} = \frac{15}{3U(M - 4)^2} \text{ cm/second.}$$