8. [10 points] Farmer Fred is designing a fence next to his barn for his grass-fed herd of cattle. The fence will be rectangular in shape with wooden fence on three sides and a chain link fence on the side closest to his barn. The wooden fence costs \$6 per foot and the chain link fence costs \$3 per foot. If he wants the fenced area to be 40,000 square feet, what should the dimensions of his fence be in order to minimize his total cost?

Solution: Let x be the length in feet of the fence along the the side closest to the barn (there is one side of this length made of wood and another made of chain link). Let y be the length in feet of two other sides, both of which are wood.
The area of the fence is $40000 = xy$. The cost of the fence is $C = 3x+6x+6y+6y = 9x+12y$. We can use the area equation to solve for x in terms of y.

$$x = 40000/y.$$

Then

$$C = 360000/y + 12y.$$

Now we calculate

$$\frac{dC}{dy} = -360000y^{-2} + 12.$$

Setting this expression equal to zero, we see that $y = \sqrt{30000}$ which makes $x = 40000/\sqrt{30000}$. To see that this is a minimum, we calculate

$$\frac{d^2C}{dy^2} = 720000y^{-3}$$

which is positive at the critical point we found above, showing that this critical point is a minimum of C. This minimum must be global since it's the only critical point on the domain of C. Therefore the dimensions which minimize the cost of the fence are $y = \sqrt{30000}$ ft by $x = 40000/\sqrt{30000}$ ft.