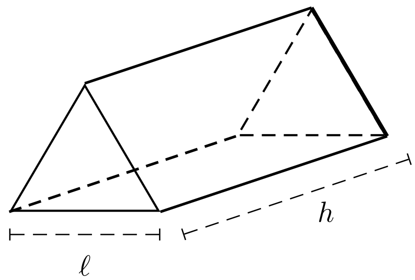
3. [12 points] Consider the prism with equilateral triangles of side length ℓ centimeters for ends and a length of h centimeters, illustrated below. The volume of this prism is $\sqrt{3}\,\ell^2 h/4$. You may find it useful to note that the area of an equilateral triangle of side length ℓ is $\sqrt{3}\,\ell^2/4$.



a. [4 points] Give the equation of the surface area of this prism, listing units.

Solution: To find the surface area, we must add up the area of all the faces, the two ends and the three rectangular sides. Therefore, Surface area $= \sqrt{3}/2\ell^2 + 3\ell h$ square centimeters.

Surface area= $\sqrt{3}/2\ell^2 + 3\ell h$ **cm**^{2}

b. [8 points] If the prism has a fixed volume of 16 cm^3, find the values of ℓ and h which minimize the surface area. Clearly justify that you have found the minimum.

Solution: We can proceed in one of two ways; we can either use the formula for volume to solve for h in terms of ℓ or ℓ in terms of h. Since volume is linear in h, it is easier to solve for h in terms of ℓ as $h = 64/\sqrt{3}\ell^{-2}$. When we plug this into the surface area equation, we get

$$S(\ell) = \frac{\sqrt{3}\,\ell^2}{2} + \frac{3(64)}{\ell\sqrt{3}}.$$

To find the minimum, we first compute the critical points of this equation:

$$\begin{aligned} S'(\ell) &= \sqrt{3}\,\ell - \frac{\sqrt{3}(64)}{\ell^2} \\ S'(\ell) = 0 \quad \Rightarrow \quad & \sqrt{3}\,\ell - \frac{\sqrt{3}(64)}{\ell^2} = 0 \\ & \sqrt{3}\,\ell = \frac{\sqrt{3}(64)}{\ell^2} \\ & \ell^3 = 64 \Rightarrow \ell = 4. \end{aligned}$$

To see that this critical point is a minimum, consider the second derivative. Since $S''(\ell) = \sqrt{3} + \frac{2\sqrt{3}(64)}{\ell^3} > 0$ for all all positive ℓ, the function $S(\ell)$ is concave up, and thus $\ell = 4$ is the local and global minimum. Therefore, the dimension of the equilateral prism with volume 16 cm^3 and minimal surface area is $\ell = 4$ cm and $h = 4/\sqrt{3}$ cm