4. [10 points] The cable of a suspension bridge with two supports $2L$ meters apart hangs H meters above the ground. The height H is given in terms of the distance in meters from the first support x (in meters) by the function

$$H(x) = e^{x-L} + e^{L-x} + H_0 - 2$$

where H_0 and L are positive constants. Notice that x ranges from 0 (the first support) to $2L$ (the second support).

a. [4 points] Find (but do not classify) the critical points for the function $H(x)$.

Solution: To find the critical points, we first take the derivative of $H(x)$:

$$H'(x) = e^{x-L} - e^{L-x}.$$

To find the critical points, we set $H'(x) = 0$:

$$\begin{aligned} H'(x) = 0 \quad &\Rightarrow \quad e^{x-L} - e^{L-x} = 0 \Rightarrow e^{x-L} = e^{L-x} \\ \text{apply } \ln \text{ to both sides} \quad & \quad x - L = L - x \\ & \quad 2x = 2L \Rightarrow x = L \end{aligned}$$

So, $H(x)$ has only one critical point at $x = L$.

b. [6 points] Find the x and y coordinates of all global maxima and minima for the function $H(x)$. Justify your answers.

Solution: Since the values of x lie in the closed interval $[0, 2L]$, to find all global maxima and minima, we need to compare the values of H at the endpoints and at any critical points. From part (a), we know the only critical point is at $x = L$, we plug $x = 0, L$, and $2L$ into $H(x)$:

$$H(0) = e^{-L} + e^{L} + H_0 - 2,\ H(L) = 1 + 1 + H_0 - 2 = H_0, \text{ and } H(2L) = e^{L} + e^{-L} + H_0 - 2.$$

To identify which of these should be larger, we notice that $e^x + e^{-x} \geq 2$ for all x. Therefore, $H(2L) = H(0) > H(L)$. Then, the function $H(x)$ has global maxima at $(0, e^{-L} + e^{L} + H_0 - 2)$ and $(2L, e^{-L} + e^{L} + H_0 - 2)$ and a global minimum at (L, H_0).