9. [12 points] Suppose $w(x)$ is an everywhere differentiable function which satisfies the following conditions:

- $w'(0) = 0$.
- $w'(x) > 0$ for $x > 0$.
- $w'(x) < 0$ for $x < 0$.

Let $f(t) = t^2 + bt + c$ where b and c are positive constants with $b^2 > 4c$. Define $L(t) = w(f(t))$.

a. [2 points] Compute $L'(t)$. Your answer may involve w and/or w' and constants b and c.

Solution: $L'(t) = w'(t^2 + bt + c) \cdot (2t + b)$.

b. [4 points] Using your answer from (a), find the critical points of $L(t)$ in terms of the constants b and c.

Solution: $L(t)$ has critical points when $L'(t) = 0$. This happens only if $w'(t^2+bt+c) = 0$ or if $(2t + b) = 0$.
$w'(t^2 + bt + c) = 0$ means $t^2 + bt + c = 0$ by the first property of w' above. Solving using the quadratic formula, we have

$$t = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

as critical points of $L(t)$. Both of these roots exist and are distinct since $b^2 > 4c$.
If $2t + b = 0$, we have $t = -\frac{b}{2}$ as a critical point. Altogether our critical points are $t = -\frac{b}{2}, -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}, -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$.

c. [6 points] Classify each critical point you found in (b). Be sure to fully justify your answer.

Solution: For simplicity, let's set $p = -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}$ and $m = -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$.

We know that $f(t)$ is an upward opening parabola with roots at p and m. We also know $p > m$, so this means $f(t) > 0$ for $t < m$ and $t > p$. This also means $f(t) < 0$ for $m < t < p$. Thus by properties two and three of w' above we know $w'(f(t)) > 0$ for $t < m$ and $t > p$, and $w'(f(t)) < 0$ for $m < t < p$.

The expression $2t + b$ is positive for $t > -\frac{b}{2}$ and negative for $t < -\frac{b}{2}$.

Putting all of this information together gives us

$$L'(t) > 0$$

on the intervals $(m, -\frac{b}{2})$ and $(p, +\infty)$, and

$$L'(t) < 0$$

on the intervals $(-\infty, m)$ and $(-\frac{b}{2}, p)$. Thus, by the first derivative test, the critical points $t = m = -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$ and $t = p = -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}$ are local minima, and $t = -\frac{b}{2}$ is a local maximum.