9. [12 points] Suppose \( w(x) \) is an everywhere differentiable function which satisfies the following conditions:

- \( w'(0) = 0 \).
- \( w'(x) > 0 \) for \( x > 0 \).
- \( w'(x) < 0 \) for \( x < 0 \).

Let \( f(t) = t^2 + bt + c \) where \( b \) and \( c \) are positive constants with \( b^2 > 4c \). Define \( L(t) = w(f(t)) \).

a. [2 points] Compute \( L'(t) \). Your answer may involve \( w \) and/or \( w' \) and constants \( b \) and \( c \).

Solution: \( L'(t) = w'(t^2 + bt + c) \cdot (2t + b) \).

b. [4 points] Using your answer from (a), find the critical points of \( L(t) \) in terms of the constants \( b \) and \( c \).

Solution: \( L(t) \) has critical points when \( L'(t) = 0 \). This happens only if \( w'(t^2 + bt + c) = 0 \) or if \( (2t + b) = 0 \).

\( w'(t^2 + bt + c) = 0 \) means \( t^2 + bt + c = 0 \) by the first property of \( w' \) above. Solving using the quadratic formula, we have

\[ t = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} \]

as critical points of \( L(t) \). Both of these roots exist and are distinct since \( b^2 > 4c \).

If \( 2t + b = 0 \), we have \( t = -\frac{b}{2} \) as a critical point. Altogether our critical points are \( t = -\frac{b}{2}, -\frac{b}{2} + \frac{\sqrt{b^2 - 4c}}{2}, -\frac{b}{2} - \frac{\sqrt{b^2 - 4c}}{2} \).

c. [6 points] Classify each critical point you found in (b). Be sure to fully justify your answer.

Solution: For simplicity, let’s set \( p = -\frac{b}{2} + \frac{\sqrt{b^2 - 4c}}{2} \) and \( m = -\frac{b}{2} - \frac{\sqrt{b^2 - 4c}}{2} \).

We know that \( f(t) \) is an upward opening parabola with roots at \( p \) and \( m \). We also know \( p > m \), so this means \( f(t) > 0 \) for \( t < m \) and \( t > p \). This also means \( f(t) < 0 \) for \( m < t < p \). Thus by properties two and three of \( w' \) above we know \( w'(f(t)) > 0 \) for \( t < m \) and \( t > p \), and \( w'(f(t)) < 0 \) for \( m < t < p \).

The expression \( 2t + b \) is positive for \( t > -\frac{b}{2} \) and negative for \( t < -\frac{b}{2} \).

Putting all of this information together gives us

\[ L'(t) > 0 \]

on the intervals \( (m, -\frac{b}{2}) \) and \((p, +\infty)\), and

\[ L'(t) < 0 \]

on the intervals \((-\infty, m)\) and \((-\frac{b}{2}, p)\). Thus, by the first derivative test, the critical points \( t = m = -\frac{b}{2} - \frac{\sqrt{b^2 - 4c}}{2} \) and \( t = p = -\frac{b}{2} + \frac{\sqrt{b^2 - 4c}}{2} \) are local minima, and \( t = -\frac{b}{2} \) is a local maximum.