4. [13 points] Let \( f(x) = e^{\sin \sqrt{x}} \). Let \( P \) be the point on the graph of \( f \) at which \( x = 4\pi^2 \approx 39.4784 \).

   a. [3 points] Calculate \( f'(x) \).
   \[
   f'(x) = \left( e^{\sin \sqrt{x}} \right) \left( \cos \sqrt{x} \right) \frac{1}{2} x^{-1/2} = \frac{e^{\sin \sqrt{x}} \cos \sqrt{x}}{2\sqrt{x}}
   \]

   b. [4 points] Find an exact formula for the tangent line \( L(x) \) to \( f(x) \) at \( P \). Exact means your answer should not involve any decimal approximations.
   \[
   \text{Slope} = f'(4\pi^2) = \frac{e^{\sin(2\pi)} \cos(2\pi)}{2 \cdot 2\pi} = \frac{1}{4\pi},
   \]
   so \( L(x) = \frac{x}{4\pi} + b \), where \( b \) is the vertical intercept. When \( f(4\pi^2) = e^{\sin(2\pi)} = 1 \), so 
   \[
   1 = \frac{4\pi^2}{4\pi} + b, \quad \text{which gives } b = 1 - \pi, \quad \text{so}
   \]
   \[
   L(x) = \frac{x}{4\pi} + 1 - \pi
   \]

   c. [2 points] Use your formula for \( L(x) \) to approximate \( e^{\sin \sqrt{38}} \).
   \[
   e^{\sin \sqrt{38}} = f(38) \approx L(38) = \frac{38}{4\pi} + 1 - \pi \approx 0.8824.
   \]

   d. [4 points] Recall that the error, \( E(x) \), is the actual value of the function minus the value approximated by the tangent line. Given the fact that in this case \( E(39) \approx 0.000613 \) and \( E(40) \approx 0.000719 \), would you expect \( f''(4\pi^2) \) to be positive or negative? Explain, without doing any calculations.
   \[
   \text{Solution: } \text{The errors are positive, which means that near } P \text{ the tangent line lies below the curve, so the function is probably concave up at } P. \text{ Since concave up corresponds to positive second derivative, we should expect the sign of } f''(4\pi^2) \text{ to be positive.}
   \]