4. [13 points] Let $f(x) = e^{\sin\sqrt{x}}$. Let P be the point on the graph of f at which $x = 4\pi^2 (\approx 39.4784)$.

a. [3 points] Calculate $f'(x)$.

Solution:

$$f'(x) = \left(e^{\sin\sqrt{x}}\right)\left(\cos\sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right) = \frac{e^{\sin\sqrt{x}}\cos\sqrt{x}}{2\sqrt{x}}$$

b. [4 points] Find an **exact** formula for the tangent line $L(x)$ to $f(x)$ at P. **Exact** means your answer should not involve any decimal approximations.

Solution:

$$\text{slope} = f'(4\pi^2) = \frac{e^{\sin(2\pi)}\cos(2\pi)}{2\cdot 2\pi} = \frac{1}{4\pi},$$

so $L(x) = \frac{x}{4\pi} + b$, where b is the vertical intercept. When $f(4\pi^2) = e^{sin(2\pi)} = 1$, so $1 = \frac{4\pi^2}{4\pi} + b$, which gives us $b = 1 - \pi$, so

$$L(x) = \frac{x}{4\pi} + 1 - \pi$$

.

c. [2 points] Use your formula for $L(x)$ to approximate $e^{\sin\sqrt{38}}$.

Solution:

$$e^{\sin\sqrt{38}} = f(38) \approx L(38) = \frac{38}{4\pi} + 1 - \pi \approx 0.8824.$$

d. [4 points] Recall that the error, $E(x)$, is the actual value of the function minus the value approximated by the tangent line. Given the fact that in this case $E(39) \approx 0.000613$ and $E(40) \approx 0.000719$, would you expect $f''(4\pi^2)$ to be positive or negative? Explain, without doing any calculations.

Solution: The errors are positive, which means that near P the tangent line lies below the curve, so the function is probably concave up at P. Since concave up corresponds to positive second derivative, we should expect the sign of $f''(4\pi^2)$ to be positive.