7. [13 points] Consider the family of functions

\[ y = ax^b \ln x \]

where \( a \) and \( b \) are nonzero constants.

a. [4 points] Calculate \( \frac{dy}{dx} \) in terms of the constants \( a \) and \( b \).

Solution: \( \frac{dy}{dx} = abx^{b-1} \ln x + ax^b \left( \frac{1}{x} \right) = abx^{b-1} \ln x + ax^{b-1} = ax^{b-1}(b \ln x + 1) \).

b. [9 points] Find specific values of \( a \) and \( b \) so that the resulting function has a local maximum at the point \((e, 1)\). You must show that \((e, 1)\) is a local maximum to receive full credit.

Solution: The point \((e, 1)\) must be on the curve, so \(1 = ae^b \ln e = ae^b\). So \(a = e^{-b}\).

We also know \((e, 1)\) is a critical point, so

\[ 0 = ae^{b-1}(b \ln e + 1) = ae^{b-1}(b-1). \]

\(a \neq 0\) and \(e^{b-1} \neq 0\) so \(b + 1 = 0\), which means \(b = -1\) and \(a = e^{-(1)} = e\).

To check that \((e, 1)\) is a local maximum, we use the second derivative test: First, we plug in the values of \(a\) and \(b\) to our derivative and get \( \frac{dy}{dx} = e^{-2}(1 - \ln x) \). So we have

\[ \frac{d^2y}{dx^2} = e(-2)x^{-3}(1 - \ln x) + e^{-2} \left( \frac{-1}{x} \right). \]

Now we can plug in \(x = e\) and get \(-2e \cdot e^{-3}(1 - \ln e) - e \cdot e^{-3} = -e^{-2} < 0\). Thus, \((e, 1)\) is a local maximum.