7. [13 points] Consider the family of functions

$$y = ax^b \ln x$$

where a and b are nonzero constants.

a. [4 points] Calculate $\frac{dy}{dx}$ in terms of the constants a and b.

Solution: $\frac{dy}{dx} = abx^{b-1}\ln x + ax^b\left(\frac{1}{x}\right) = abx^{b-1}\ln x + ax^{b-1} = ax^{b-1}(b\ln x + 1)$.

b. [9 points] Find specific values of a and b so that the resulting function has a local maximum at the point $(e, 1)$. You must show that $(e, 1)$ is a local maximum to receive full credit.

Solution: The point $(e, 1)$ must be on the curve, so $1 = ae^b \ln e = ae^b$. So $a = e^{-b}$. We also know $(e, 1)$ is a critical point, so

$$0 = ae^{b-1}(b\ln e + 1) = ae^{b-1}(b - 1).$$

$a \neq 0$ and $e^{b-1} \neq 0$ so $b + 1 = 0$, which means $b = -1$ and $a = e^{-(-1)} = e$.
To check that $(e, 1)$ is a local maximum, we use the second derivative test:
First, we plug in the values of a and b to our derivative and get $\frac{dy}{dx} = ex^{-2}(1 - \ln x)$. So we have

$$\frac{d^2y}{dx^2} = e(-2)x^{-3}(1 - \ln x) + ex^{-2}\left(\frac{-1}{x}\right).$$

Now we can plug in $x = e$ and get $-2e \cdot e^{-3}(1 - \ln e) - e \cdot e^{-3} = -e^{-2} < 0$. Thus, $(e, 1)$ is a local maximum.