8. [13 points] Two smokestacks d miles apart deposit soot on the ground between them. The concentration of the combined soot deposits on the line joining them, at a distance x from one stack, is given by

$$S = \frac{c}{x^2} + \frac{k}{(d-x)^2}$$

where c and k are positive constants which depend on the quantity of smoke each stack is emitting. If $k = 27c$, find the x-value of the point on the line joining the stacks where the concentration of the deposit is a minimum. Justify that the point you found is actually a global minimum.

Solution: First we plug in $k = 27c$ and get

$$S = \frac{c}{x^2} + \frac{27c}{(d-x)^2}.$$

Then we take the derivative and set it equal to zero to find the critical points on the domain $0 < x < d$:

$$S' = \frac{-2c}{x^3} + \frac{(-2)(27c)(-1)}{(d-x)^3} = \frac{-2c}{x^3} + \frac{2 \cdot 27c}{(d-x)^3} = 0$$

Now we solve for x:

$$\frac{2 \cdot 27c}{(d-x)^3} = \frac{2c}{x^3}$$

$$\frac{27}{(d-x)^3} = \frac{1}{x^3}$$

Taking the cube root of both sides gives $\frac{3}{d-x} = \frac{1}{x}$. So $3x = d - x$. Thus $x = \frac{d}{4}$.
Now our domain is $(0, d)$ and $x = \frac{d}{4}$ is our only critical point. As $x \to 0$, $S \to \infty$, and as $x \to d$, $S \to \infty$. Thus, our global minimum must be at our sole critical point at $x = \frac{d}{4}$.