9. [13 points] Consider the function

$$f(x) = ax \ln x - bx$$

with domain $x > 0$, where a and b are positive constants. Note that this function has exactly one critical point.

a. [3 points] Find $f'(x)$.

Solution:

$$f'(x) = a\left((1)\ln x + x\frac{1}{x}\right) - b = a\ln x + a - b$$

b. [4 points] For which values of a and b does $f(x)$ have a critical point at $(e, -2)$?

Solution: We want $f'(e) = 0$ and $f(e) = -2$. We plug $x = e$ into $f'(x)$ to get

$$f'(e) = a\ln e + a - b = 2a - b = 0$$

which tells us that $2a = b$. Plugging $x = e$ into $f(x)$ gives

$$f(e) = ae\ln e - be = (a - b)e = -2.$$

Using $2a = b$, we get

$$(a - (2a))e = -2$$

so that $a = 2/e$. Since $b = 2a$, we get $b = 4/e$.

c. [3 points] Using your values of a and b from part **(b)**, is the critical point from **(b)** a local maximum, local minimum, or neither? Justify your answer.

Solution: The second derivative of $f(x)$ is $f''(x) = a/x = (4/e)/x$. Then $f''(e) = 4/e^2 > 0$ so the graph of $f(x)$ is concave up at $x = e$. This means that $f(x)$ has a local minimum at $x = e$.

d. [3 points] Using your values of a and b from part **(b)**, find the x-coordinates of any inflection points of $f(x)$ or show that $f(x)$ has no inflection points.

Solution: The second derivative of $f(x)$ is $f''(x) = a/x = (4/e)/x$. This is continuous and positive for all values of x in the domain $x > 0$ of $f(x)$. Since the second derivative never changes sign, $f(x)$ has no inflection points.