9. [13 points] Consider the function 

\[ f(x) = ax \ln x - bx \]

with domain \( x > 0 \), where \( a \) and \( b \) are positive constants. Note that this function has exactly one critical point.

a. [3 points] Find \( f'(x) \).

\[ f'(x) = a \left( 1 + \frac{1}{x} \right) - b = a \ln x + a - b \]

Solution:

b. [4 points] For which values of \( a \) and \( b \) does \( f(x) \) have a critical point at \((e, -2)\)?

\[ f'(e) = a \ln e + a - b = 2a - b = 0 \]

\[ f(e) = a e \ln e - be = (a - b)e = -2 \]

Using \( 2a = b \), we get \( (a - 2a)e = -2 \) so that \( a = 2/e \). Since \( b = 2a \), we get \( b = 4/e \).

Solution:

The second derivative of \( f(x) \) is \( f''(x) = a/x = (4/e)/x \). Then \( f''(e) = 4/e^2 > 0 \) so the graph of \( f(x) \) is concave up at \( x = e \). This means that \( f(x) \) has a local minimum at \( x = e \).

Solution:

The second derivative of \( f(x) \) is \( f''(x) = a/x = (4/e)/x \). This is continuous and positive for all values of \( x \) in the domain \( x > 0 \) of \( f(x) \). Since the second derivative never changes sign, \( f(x) \) has no inflection points.