8. [10 points] The citizens of Srebmun Foyoj have decided to put a bed of mumertxe flowers in their new park. The floral density D (in flowers per square meter) of a flowerbed of area A square meters is given by $D = f(A)$. Formulas for $f(A)$ and its derivative $f'(A)$ are given below.

$$f(A) = 30\left(\frac{A^3 - 4.5A^2 + 4.5A - 0.5}{e^A}\right) + 15, \text{ and } f'(A) = -30\left(\frac{(A-0.5)(A-2)(A-5)}{e^A}\right).$$

a. [5 points] The citizens intend to make the area of the flowerbed between 1.5 and 3.5 square meters. What area A (with $1.5 \le A \le 3.5$) should they make the flowerbed in order to maximize the density of the flowers in the flowerbed? Use calculus to find and justify your answer, and be sure to show enough evidence to demonstrate that the area you find does indeed maximize the density of the flowers.

Solution: Since $f'(A)$ is always defined (as e^A is never zero), the critical points of f are at $A = 0.5, 2,$ and 5. Of these, only $A = 2$ is in the interval in question. Since f is continuous and our domain is a closed interval, we can apply the Extreme Value Theorem, and need only evaluate $f(A)$ at $A = 1.5, 2,$ and 3.5 and choose the value of A for which $f(A)$ is largest. We find

$$f(1.5) \approx 11.65$$
$$f(2) \approx 8.91$$
$$f(3.5) \approx 17.72.$$

Therefore $f(A)$ attains its maximum value on the interval $[1.5, 3.5]$ when $A = 3.5$. So the density of the flowers in the flowerbed will be maximized when the flowerbed has area 3.5 square meters.

Answer: Maximum density when area $A =$ 3.5

b. [5 points] Suppose instead that the citizens can make the flowerbed any area greater than or equal to 1.5 square meters. What are the largest and smallest densities this flowerbed could have? Use calculus to find your answer and be sure to show enough evidence to demonstrate that you have found the minimum and maximum densities.

Solution: From part (a), we know that $f(1.5) \approx 11.65$ and $f(2) \approx 8.91$. Since $A = 5$ is now in the domain we are considering, we need to also consider $f(5) \approx 21.98$. Together with our data from part (a) above, this implies that on the closed interval $1.5 \le A \le 5$, the minimum value of f is $f(2)$ (≈ 8.91), and the maximum value of f is $f(5)$ (≈ 21.98).

Note that for $A > 5$, the sign of $f'(A)$ is given by $- \cdot \frac{+ \cdot + \cdot +}{+}$ so $f'(A) < 0$ on this interval. Thus we know that $f(A)$ is decreasing on the interval $(5, \infty)$. Since $\lim_{A\to\infty} f(A) = 15$, we see that for $A > 5$, we have $15 < f(A) < f(5)$ (so the value of $f(A)$ is always between $f(5)$ (≈ 21.98) and 15).

Therefore, for $A \ge 1.5$, we have a maximum density of about 21.98 flowers per square meter and a minimum density of about 8.91 flowers per square meter.

(For each answer blank below, write NONE if appropriate.)

Answer: Maximum density: $D =$ 21.98

Answer: Minimum density: $D =$ 8.91