7. [10 points] Suppose $f(x)$ is a continuous function defined for all real numbers whose derivative and second derivative are given by

$$f'(x) = \arctan\left(\frac{(2x-3)^2}{(x+1)^3}\right) \qquad \text{and} \qquad f''(x) = \frac{(x+1)^3(2x-3)(13-2x)}{(x+1)\left[(x+1)^6+(2x-3)^4\right]}.$$

a. [5 points] Find all critical points of $f(x)$ and all values of x at which $f(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write NONE if appropriate.

Solution: First we find the critical points, which occur when $f'(x) = 0$ or $f'(x)$ does not exist. Since $f'(x) = 0$ when $x = \frac{3}{2}$ and $f'(x)$ is not defined when $x = -1$, the two critical points are $x = -1$ and $x = \frac{3}{2}$.
Next we must determine whether there is a local min, local max, or neither at each critical point. The second derivative test is inconclusive, because $f''(-1)$ does not exist and $f''(\frac{3}{2}) = 0$. So we must use the first derivative test. Notice that arctan preserves signs (i.e., $\arctan(x)$ is positive when x is positive and $\arctan(x)$ is negative when x is negative) so we only need to check the sign of $\left(\frac{(2x-3)^2}{(x+1)^3}\right)$. The factor $(2x-3)^2$ is always positive, while $(x+1)^3$ is negative when $x < -1$ and positive when $x > -1$. This gives us the resulting signs:

Interval	$x < -1$	$-1 < x < \frac{3}{2}$	$x > \frac{3}{2}$
Sign of $f'(x)$	$\frac{+}{-} = -$	$\frac{+}{+} = +$	$\frac{+}{+} = +$

So $f(x)$ has a local minimum at $x = -1$ and no local maxima.

Answer: Critical point(s) at $x =$ $-1, \frac{3}{2}$

Local max(es) at $x =$ **None** Local min(s) at $x =$ -1

b. [5 points] Find the x-coordinates of all inflection points of $f(x)$. If there are none, write NONE. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

Solution: First we find the candidate inflection points, which occur when $f''(x) = 0$ or $f''(x)$ does not exist. We can see that $f''(x) = 0$ at $x = \frac{3}{2}$ and $\frac{13}{2}$ and that $f''(x)$ is undefined when $x = -1$. To determine whether these are actually inflection points (where concativty changes), we must test the sign of the second derivative on either side of each of the points. We find the following:

Interval	$x < -1$	$-1 < x < \frac{3}{2}$	$\frac{3}{2} < x < \frac{13}{2}$	$x > \frac{13}{2}$
Sign of $f'(x)$	$\frac{-\cdot-\cdot+}{-\cdot+} = -$	$\frac{+\cdot-\cdot+}{+\cdot+} = -$	$\frac{+\cdot+\cdot+}{+\cdot+} = +$	$\frac{+\cdot+\cdot-}{+\cdot+} = -$

The inflection points of $f(x)$ occur at the points where the sign of the second derivative changes, that is, at $x = \frac{3}{2}$ and $x = \frac{13}{2}$.

Answer: Inflection point(s) at $x =$ $\frac{3}{2}, \frac{13}{2}$