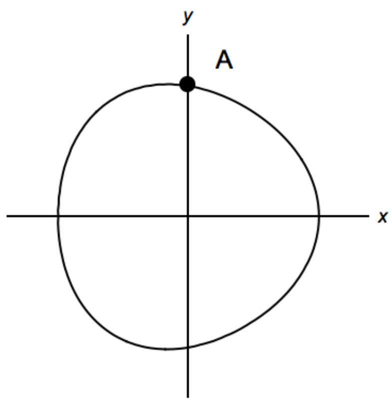
8. [11 points] Let $\mathcal{C}$ be the curve given by the equation $81 - (x^2 + y^2)^2 = 2xy^2$. The graph of $\mathcal{C}$ is shown below.



a. [2 points] Find the coordinates (x, y) of the point A.

Solution: Since the point A lies at the intersection of the y-axis and the curve $\mathcal{C}$, then $x = 0$ and y satisfies $81 - (0^2 + y^2)^2 = 2(0)xy^2$. Hence $y^4 = 81$ or $y = 3$.

$A = (0, 3)$

b. [6 points] Find $\dfrac{dy}{dx}$. Show all your computations step by step.

Solution:

$$\frac{d}{dx}\left(81 - (x^2 + y^2)^2\right) = \frac{d}{dx}\left(2xy^2\right)$$

$$-2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 2y^2 + 4xy\frac{dy}{dx}$$

$$-4x(x^2 + y^2) - 4y(x^2 + y^2)\frac{dy}{dx} = 2y^2 + 4xy\frac{dy}{dx}$$

$$-4y(x^2 + y^2)\frac{dy}{dx} - 4xy\frac{dy}{dx} = 2y^2 + 4x(x^2 + y^2)$$

$$\frac{dy}{dx} = -\frac{2y^2 + 4x(x^2 + y^2)}{4y(x^2 + y^2) + 4xy}$$

c. [3 points] Find an equation of the tangent line $L(x)$ to the graph of $\mathcal{C}$ at A. Show all your work.

Solution: The slope of $L(x)$ is

$$m = -\frac{2(3)^2 + 4(0)((0)^2 + (3)^2)}{4(3)((0)^2 + (3)^2) + 4(0)(3)} = -\frac{18}{108} = -\frac{1}{6}.$$

Hence using the point A and the slope-intercept formula for the line $L(x)$, we get $L(x) = -\dfrac{1}{6}x + 3$.