9. [7 points] Let A and B be two constants and

$$h(x) = \begin{cases} 2Bx + A\ln(x) & 0 < x \leq 1 \\ \\ \dfrac{4A}{x} + Bx - 1 & 1 < x \leq 2. \end{cases}$$

Find all the values of A and B that make the function $h(x)$ differentiable on the interval $0 < x < 2$. If no such values exist, write NONE. Justify your answer.

Solution: Since $y = 2Bx + A\ln(x)$ and $y = \frac{4A}{x} + Bx - 1$ are differentiable on the intervals $(0, 1)$ and $(1, 2)$, then we only need to choose A and B so that $h(x)$ is differentiable at $x = 1$.
In order to obtain continuity at $x = 1$, A and B must satisfy

$$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-} 2Bx + A\ln(x) = \mathbf{2B} = \mathbf{4A} + \mathbf{B} - \mathbf{1} = \lim_{x\to 1^+} \frac{4A}{x} + Bx - 1 = \lim_{x\to 1^+} h(x)$$

This equation can be simplified to $\mathbf{B} = \mathbf{4A} - \mathbf{1}$.
Since $y = 2Bx + A\ln(x)$ and $y = \frac{4A}{x} + Bx - 1$ are differentiable functions on $(0, 4)$, then differentiability of $h(x)$ at $x = 1$ follows if their derivatives $y' = 2B + \frac{A}{x}$ and $y' = -\frac{4A}{x^2} + B$ are equal at $x = 1$. This yields $\mathbf{2B} + \mathbf{A} = -\mathbf{4A} + \mathbf{B}$ or $\mathbf{B} = -\mathbf{5A}$.

Solving both equations $\mathbf{B} = -\mathbf{5A}$ and $\mathbf{B} = \mathbf{4A} - \mathbf{1}$, we get that $-5A = 4A - 1$. Therefore $\mathbf{A} = \mathbf{\frac{1}{9}}$ and then $\mathbf{B} = -\mathbf{\frac{5}{9}}$.