11. [8 points] Consider the function

$$f(x) = \begin{cases} 4 - x - x^{\frac{2}{3}} & \text{for } -8 \le x \le 0 \\ 5xe^{-0.5x} + 4 & \text{for } x > 0. \end{cases} \quad \text{and its derivative} \quad f'(x) = \begin{cases} \dfrac{2 + 3x^{\frac{1}{3}}}{-3x^{\frac{1}{3}}} & \text{for } -8 < x < 0 \\ 5(1 - 0.5x)e^{-0.5x} & \text{for } x > 0. \end{cases}$$

Find the x-coordinates of the global maximum and the global minimum of the function $f(x)$ for $x \ge -8$. If one of them does not exist, write NONE in the answer line below. Use calculus to find your answers, and be sure to show enough evidence that the point(s) you find are indeed global extrema.

Solution: Critical points on the interval $(-8, 0)$ occur when the expression $\dfrac{2 + 3x^{\frac{1}{3}}}{-3x^{\frac{1}{3}}}$ is 0 or undefined. The numerator is equal to 0 at $x = -\frac{8}{27}$. The denominator is equal to zero when $x = 0$, which is not in the interval, so the only critical point in the interval $(-8, 0)$ is at $x = -\frac{8}{27}$.

Critical points on the interval $(0, \infty)$ occur when the expression $5(1 - 0.5x)e^{-0.5x}$ is 0 or undefined. Since $e^{-0.5x}$ is always positive, the only critical point is when $1 - 0.5x = 0$, which occurs when $x = 2$.

In addition to critical points, we must also check for global extrema by checking the end behavior for each piece of $f(x)$.

$$f(-8) = 4 - (-8) - (-8)^{2/3} = 12 - 4 = 8$$

$$f\left(-\frac{8}{27}\right) = 4 - \left(-\frac{8}{27}\right) - \left(-\frac{8}{27}\right)^{2/3} = 4 + \frac{8}{27} - \frac{4}{9} = \frac{104}{27}$$

$$f(0) = 4 - 0 - 0 = 4$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (5xe^{-0.5x} + 4) = 5 \cdot 0 \cdot 1 + 4 = 4$$

$$f(2) = 5 \cdot 2 \cdot e^{-1} + 4 = \frac{10}{e} + 4$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{5x}{e^{0.5x}} + 4 = 0 + 4 = 4$$

None of the end behavior is infinite. The highest number in the calculations above is 8, which occurs at $x = -8$. The lowest number is $\dfrac{104}{27}$, which occurs at $x = -\dfrac{8}{27}$

Answer:

Global maximum(s) at $x =$ -8

Global minimum(s) at $x =$ $-\frac{8}{27}$