2. [6 points] Let A and B be constants and

$$k(x) = \begin{cases} 3x + \dfrac{B}{x} & \text{for} \quad 0 < x < 1 \\ Bx^2 + Ax^3 & \text{for} \quad 1 \le x. \end{cases}$$

Find the values of A and B that make the function $k(x)$ differentiable on $(0, \infty)$. Show all your work to justify your answers. If there are no such values of A and B, write NONE.

Solution: $k(x)$ will only be differentiable at $x = 1$ if it is also continuous at $x = 1$. In order for this to happen, we plug $x = 1$ in to both formulas of the original function and set them equal as well:

$$3 + B = B + A$$

From this second equation, we can subtract B from both sides to find $A = 3$.
The function $3x + \frac{B}{x}$ is differentiable on $(0, 1)$, and the function $Bx^2 + Ax^3$ is differentiable on $(1, \infty)$, so we just need values of A and B that will make $k(x)$ differentiable at $x = 1$.
We can compute the derivative:

$$k'(x) = \begin{cases} 3 - \dfrac{B}{x^2} & \text{for} \quad 0 < x < 1 \\ 2Bx + 3Ax^2 & \text{for} \quad 1 < x. \end{cases}$$

In order for $k(x)$ to be differentiable at $x = 1$, we must have

$$\begin{aligned} \lim_{h \to 0^-} \frac{k(1+h) - k(1)}{h} &= \lim_{h \to 0^+} \frac{k(1+h) - k(1)}{h} \\ \frac{d}{dx}\left(3x + \frac{B}{x}\right)\Big|_{x=1} &= \frac{d}{dx}(Bx^2 + Ax^3)\Big|_{x=1} \\ 3 - \frac{B}{x^2}\Big|_{x=1} &= 2Bx + 3Ax^2\big|_{x=1} \\ 3 - B &= 2B + 3A \end{aligned}$$

In addition, Now we plug this value in for A in the earlier equation, giving us

$$3 - B = 2B + 9$$

Solving for B, we get $3B = -6$, so $B = -2$.

Answer: $A = 3$ $\qquad B = -2$