8. [7 points] Yi is working on a second box (not the one from the previous problem). This box will need a wire frame. The base of the box will be a square with width w inches. The height of the box will be h inches. Once the box is completed, its volume, V, in cubic inches will be

$$V = w^2 h.$$

Yi has to use thicker wire for the edges along the top and bottom of the box. Let M be the total mass in grams of the wire frame. The equation for M is

$$M = 18w + 6h.$$

Yi has a total of 540 grams of metal to make the wire frame. What values of w and h will maximize the volume of his box? Use calculus to find and justify your answer, and be sure to show enough evidence that the values you find do in fact maximize the volume.

Solution: Solving for w in terms of h:

$$w = 30 - \frac{1}{3}h$$

$$V = \left(30 - \frac{1}{3}h\right)^2 h$$

$$V = \frac{1}{9}h^3 - 20h^2 + 900h$$

Now find the critical points of V.

$$\frac{dV}{dh} = \frac{1}{3}h^2 - 40h + 900$$

Critical points occur when $V' = 0$ so $h = 30$ or $h = 90$. Since w and h should both be positive, we get $h > 0$ and $30 - \frac{1}{3}h > 0$ so the possible values of h satisfy $0 < h < 90$. Since V is continuous on this interval and has only one critical point, $h = 30$, we only have to test the behavior of $V = \frac{1}{9}h^3 - 20h^2 + 900$ close to the endpoints $h = 0$ and $h = 90$ and the critical point $h = 30$.

$\lim\limits_{h\to 0^+} V$	$h = 30$	$\lim\limits_{h\to 90^-} V$
0	V(30)=12,000	0

Answer: The volume of the box is maximum when $w =$ 20 $h =$ 30

Other approach in this case (after finding the critical points of V in $0 < h < 90$): In $0 < h < 30$ and $30 < h < 90$ we pick $h = 20$ and $h = 50$ to find the sign of $\frac{dV}{dh}$ in these intervals. We get

$$\frac{dV}{dh}\Big|_{h=20} \approx 233.33 \quad \textit{and} \quad \frac{dV}{dh}\Big|_{h=50} \approx -266.66$$

	$0 < h < 30$	$30 < h < 90$
$\frac{dV}{dh}$	$+$	$-$

So, $h = 30$ is local max. Since V is continuous on this interval and has only one critical point, $h = 30$ is also the global max. When $h = 30$ we get $w = 20$.