9. [13 points] A curve $C$ is defined implicitly by the equation

$$2x(y^2 - 4y) + 5x = 15.$$ 

Note that the curve $C$ satisfies

$$\frac{dy}{dx} = \frac{-2y^2 + 8y - 5}{4xy - 8x}.$$ 

a. [4 points] Find all points on $C$ with a vertical tangent line. Give your answers as ordered pairs (coordinates). Justify your answer algebraically. Write NONE if no such points exist.

**Solution:** Let’s have a vertical tangent line when the denominator of $\frac{dy}{dx}$ is 0, that is, $4xy - 8x = 0$. Factoring, we can rewrite this as $4x(y - 2) = 0$. This happens when $x = 0$ or $y = 2$.

Now we need to find points on $C$ where $x = 0$ or $y = 2$. Plugging $x = 0$ in to the equation for $C$, we get $2 \cdot 0 \cdot (y^2 - 4y) + 5 \cdot 0 = 15$, which reduces to $0 = 15$. Since this is not true for any value of $y$, there are no points $(0, y)$ satisfying this equation.

Next we plug in $y = 2$. This gives

$$2 \cdot (2^2 - 4 \cdot 2) + 5 \cdot 3 = 2 \cdot (-4) + 5 \cdot 3 = -8 + 15 = 7.$$ 

So the denominator is 0 at the point $(x, y) = (-5, 2)$.

To make sure the numerator is not also 0, we plug $y = 2$ in to the numerator:

$$-2 \cdot 2^2 + 8 \cdot 2 - 5 = -8 + 16 - 5 = 3 \neq 0.$$ 

**Answer:** $(-5, 2)$

b. [4 points] There is one point on $C$ with a coordinate $(k, 0)$. Find the value of $k$ and write the equation of the tangent line to $C$ at the point $(k, 0)$. Your equation should not include the letter $k$.

**Solution:** To find the value of $k$, we plug $(k, 0)$ into the equation for $C$.

$$2 \cdot k(0^2 - 4 \cdot 0) + 5k = 15$$ 
$$0 + 5k = 15$$ 
$$k = 3.$$ 

So $k = 3$. Now we plug $(3, 0)$ into the formula for $\frac{dy}{dx}$ to find the slope of the tangent line at this point: 

$$\frac{-2 \cdot 0 + 8 \cdot 0 - 5}{4 \cdot 3 \cdot 0 - 8 \cdot 3} = \frac{-5}{24}.$$ 

Finally, we use point-slope form to get the equation for the tangent line: 

$$y = \frac{5}{24}(x - 3).$$ 

**Answer:** $k = 3$ Equation of tangent line: $y = \frac{5}{24}(x - 3)$
c. [5 points] Another curve $\mathcal{D}$ is defined implicitly by the equation

$$\cos(Wy^3 + Vxy) = \frac{1}{6}$$

where $W$ and $V$ are constants. Find a formula for $\frac{dy}{dx}$ in terms of $x$, $y$, $W$, and $V$. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

**Solution:** Differentiating both sides with respect to $x$ and then solving for $\frac{dy}{dx}$ gives

$$-\sin(Wy^3 + Vxy)(3Wy^2 \frac{dy}{dx} + Vy + Vx \frac{dy}{dx}) = 0$$

$$\Rightarrow \quad (3Wy^2 + Vx)\frac{dy}{dx} + Vy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-Vy}{3Wy^2 + Vx}$$

Alternatively, we can start by taking arccos of both sides, giving us $Wy^3 + Vxy = \arccos\left(\frac{1}{6}\right)$, and then use implicit differentiation to get $(3Wy^2 + Vx)\frac{dy}{dx} + Vy = 0$; from here, we solve as in above.

**Answer:** $\frac{dy}{dx} = \frac{-Vy}{3Wy^2 + Vx}$