9. [13 points] A curve $\mathcal{C}$ is defined implicitly by the equation

$$2x(y^2 - 4y) + 5x = 15.$$

Note that the curve $\mathcal{C}$ satisfies

$$\frac{dy}{dx} = \frac{-2y^2 + 8y - 5}{4xy - 8x}.$$

a. [4 points] Find all points on $\mathcal{C}$ with a vertical tangent line. Give your answers as ordered pairs (coordinates). Justify your answer algebraically. Write NONE if no such points exist.

Solution: $\mathcal{C}$ will have a vertical tangent line when the denominator of $\frac{dy}{dx}$ is 0, that is, $4xy - 8x = 0$. Factoring, we can rewrite this as $4x(y-2) = 0$. This happens when $x = 0$ or $y = 2$.
Now we need to find points on $\mathcal{C}$ where $x = 0$ or $y = 2$. Plugging $x = 0$ in to the equation for $\mathcal{C}$, we get $2 \cdot 0 \cdot (y^2 - 4y) + 5 \cdot 0 = 15$, which reduces to $0 = 15$. Since this is not true for any value of y, there are no points $(0, y)$ satisfying this equation.
Next we plug in $y = 2$. This gives

$$\begin{aligned} 2x \cdot (2^2 - 4 \cdot 2) + 5x &= 15 \\ 2x \cdot (-4) + 5x &= 15 \\ -3x &= 15 \\ x &= -5 \end{aligned}$$

So the denominator is 0 at the point $(x, y) = (-5, 2)$.
To make sure the numerator is not also 0, we plug $y = 2$ in to the numerator:

$$-2 \cdot 2^2 + 8 \cdot 2 - 5 = -8 + 16 - 5 = 3 \neq 0$$

Answer: $(-5, 2)$

b. [4 points] There is one point on $\mathcal{C}$ with a coordinate $(k, 0)$. Find the value of k and write the equation of the tangent line to $\mathcal{C}$ at the point $(k, 0)$. Your equation should not include the letter k.

Solution: To find the value of k, we plug $(k, 0)$ into the equation for $\mathcal{C}$.

$$\begin{aligned} 2 \cdot k(0^2 - 4 \cdot 0) + 5k &= 15 \\ 0 + 5k &= 15 \\ k &= 3 \end{aligned}$$

So $k = 3$. Now we plug $(3, 0)$ into the formula for $\frac{dy}{dx}$ to find the slope of the tangent line at this point: $\frac{-2 \cdot 0 + 8 \cdot 0 - 5}{4 \cdot 3 \cdot 0 - 8 \cdot 3} = \frac{5}{24}$. Finally, we use point-slope form to get the equation for the tangent line: $y = \frac{5}{24}(x - 3)$.

Answer: $k = 3$ Equation of tangent line: $y = \dfrac{5}{24}(x - 3)$

c. [5 points] Another curve $\mathcal{D}$ is defined implicitly by the equation

$$\cos(Wy^3 + Vxy) = \frac{1}{6}$$

where W and V are constants. Find a formula for $\frac{dy}{dx}$ in terms of x, y, W, and V. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution: Differentiating both sides with respect to x and then solving for $\frac{dy}{dx}$ gives

$$\begin{aligned} -\sin(Wy^3 + Vxy)(3Wy^2\frac{dy}{dx} + Vy + Vx\frac{dy}{dx}) &= 0 \\ (3Wy^2 + Vx)\frac{dy}{dx} + Vy &= 0 \\ \frac{dy}{dx} &= \frac{-Vy}{3Wy^2 + Vx} \end{aligned}$$

Alternatively, we can start by taking arccos of both sides, giving us $Wy^3 + Vxy = \arccos(\frac{1}{6})$, and then use implicit differentiation to get $(3Wy^2 + Vx)\frac{dy}{dx} + Vy = 0$; from here, we solve as in above.

Answer: $\frac{dy}{dx} = \frac{-Vy}{3Wy^2 + Vx}$