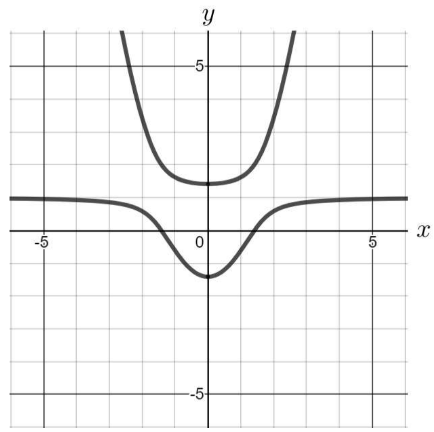
9. [6 points]
The implicit curve $\mathcal{C}$ is given by the equation

$$y^2 - 1 = r^2 + x^2(y - r)$$

for some constant r. A graph of the curve with $r = 1$ is shown to the right. Note that

$$\frac{dy}{dx} = \frac{2x(y-r)}{2y - x^2}.$$

Answer each of the following questions about the implicit curve $\mathcal{C}$. Your answers must be in **exact form**.



a. [2 points] When $r = 1$, the curve $\mathcal{C}$ passes through the point $(\sqrt{2}, 0)$. Write a formula for the tangent line to the curve $\mathcal{C}$ at this point.

Solution: The slope at $(\sqrt{2}, 0)$ when $r = 1$ is

$$\frac{2\sqrt{2}(0-1)}{2(0) - (\sqrt{2})^2} = \frac{2\sqrt{2}}{2} = \sqrt{2}.$$

Answer: $y = \sqrt{2}\,(x - \sqrt{2})$

b. [4 points] In this part, we do not assume anything about r. In particular, do <u>not</u> assume $r = 1$. Find the (x, y) coordinates of **<u>all</u>** points at which the tangent line to the curve $\mathcal{C}$ is horizontal. If there are no such points, write NONE. Your answer may be in terms of the constant r. You must show every step of your work.

Solution: For the tangent line to $\mathcal{C}$ at (x, y) to be horizontal, we need the numerator of dy/dx to equal zero:

$$2x(y - r) = 0, \quad \text{meaning} \quad x = 0 \quad \text{or} \quad y = r.$$

If there was a point on the curve with $y = r$, then we would have the equation

$$\begin{aligned} r^2 - 1 &= r^2 + x^2(r - r) \\ r^2 - 1 &= r^2 \\ -1 &= 0, \end{aligned}$$

so there are no points with $y = r$. To find points with $x = 0$, we solve

$$\begin{aligned} y^2 - 1 &= r^2 + (0)^2(y - r) \\ y^2 &= r^2 + 1 \\ y &= \pm\sqrt{r^2 + 1}. \end{aligned}$$

Answer: $\left(0, \sqrt{r^2+1}\right)$ **and** $\left(0, -\sqrt{r^2+1}\right)$