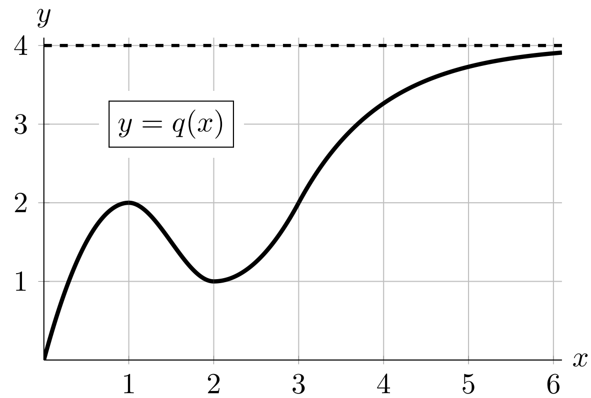
8. [10 points]

The graph of a function $q(x)$ is shown on the right. Note that:

- the domain of $q(x)$ is $[0, \infty)$,
- $q(x)$ has critical points at $x = 1$ and $x = 2$,
- $q(x)$ has no critical points for $x \geq 4$, and
- $q(x)$ has a horizontal asymptote at $y = 4$.



Now consider the piecewise-defined function $r(x)$ given as follows, where $q(x)$ is as given above:

$$r(x) = \begin{cases} \frac{1}{2}x^3 - \frac{3}{2}x & \text{if } x \leq 0 \\ q(x) & \text{if } x > 0 \end{cases}.$$

For each part below, you must use calculus to find and justify your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

a. [5 points] Find the x-coordinates of

i. the global minimum(s) of $r(x)$ on $[-1, 3]$ and

ii. the global maximum(s) of $r(x)$ on $[-1, 3]$.

If there are none of a particular type, write NONE.

Solution: The derivative of $\frac{1}{2}x^3 - \frac{3}{2}x$ is

$$\frac{3}{2}x^2 - \frac{3}{2} = \frac{3}{2}(x-1)(x+1),$$

so $r(x)$ has critical points at $x = -1$ for the left piece, and 1 and 2 from the right piece. Since $0^3 - 3(0) = 0 = q(0)$, the function r is continuous and we can use the extreme value theorem. But, note that $x = 0$ is a critical point because there is a corner there, since the slope from the left approaches $-\frac{3}{2}$, while the slope on the right side is clearly positive. To determine which of our critical points and endpoints might be global extrema, we make a table of values.

x	-1	0	1	2	3
$r(x)$	1	0	2	1	2

Hence a global minimum occurs at $x = 0$ and global maxima occur at $x = 1$ and 3.

b. [5 points] Find the x-coordinates of

i. the global minimum(s) of $r(x)$ on $(-\infty, \infty)$ and

ii. the global maximum(s) of $r(x)$ on $(-\infty, \infty)$.

If there are none of a particular type, write NONE.

Solution: In the previous part we examined all the critical points of $r(x)$, so in this part we need only examine the end behavior. Since

$$\lim_{x \to -\infty} r(x) = -\infty \qquad \text{and} \qquad \lim_{x \to \infty} r(x) = 4,$$

there are no global maxima or minima on this interval: both answers are NONE.